

Entrepreneurs, Risk Aversion and Dynamic Firms

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February 28, 2014

Abstract

This paper conducts a theoretical and quantitative analysis of how entrepreneurs choose firm size, capital structure, default, and owner consumption to manage firm risk, including how these choices change with risk aversion. We decompose an entrepreneur's default decision into three elements: the fraction of firm debt; the potential reduction in personal consumption from losing the firm; and the ratio of personal wealth to firm scale, which determines an entrepreneur's ability to inject personal funds to continue operation. Data from the Survey of Small Business Finances is used to calibrate the model and estimate entrepreneur risk aversion. We determine the evolution of entrepreneur net worth, consumption, and firm assets over time. We find that many entrepreneurs have lower net worth and consumption than non-entrepreneurs with the same preferences, but the densities of the distributions of consumption and net-worth have wide upper tails. Thus, entrepreneurship can be a path toward great wealth and high consumption for the top quantiles of entrepreneurs.

JEL Classification Numbers: D92, E01, G33, G38, L25, L26

Keywords: Entrepreneur; Risk Aversion; Endogenous Default; Credit Constraints; Financial Structure; Heterogeneity; SSBF

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We thank Marco Cagetti, George Deltas, Tim Kehoe, Makoto Nakajima, Stephen Parente, Vincenzo Quadrini, Constantine Yannelis and especially discussants Cristina De Nardi, Jamsheed Shorish, Yiannis Vailakis, two referees, and an editor. We also thank seminar participants at Cambridge, Oxford, University of London, the conferences on Bankruptcy at the IHS in Vienna, GE at Warwick University, SAET, SWET and the Milton Friedman Institute-Chicago FED Conference on Finance and Development. We gratefully acknowledge financial support from National Science Foundation grant SES-031839, NCSA computation grant SES050001, the Academy for Entrepreneurial Leadership at the University of Illinois and Kauffman Foundation grant 20061258. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation or any other organization.

1 Introduction

Willingness to take risks has a long tradition as an important characteristic of successful entrepreneurs, see Cantillon (1755). One view suggests that entrepreneurs may be less risk averse than the population at large, e.g., Knight (1921) and Kihlstrom and Laffont (1979). An alternative view is that entrepreneurs are particularly skilled at managing their exposure to risk. For example, Paul Brown in *Forbes* reports that entrepreneurs “don’t like risk. They accept it as part of the game and then work extremely hard to reduce it to a minimum.”¹ We evaluate these views in a dynamic model in which entrepreneurs manage risk by selecting firm size, capital structure, default, and their own consumption. Individuals are risk averse, have access to a constant returns to scale production technology, and are endowed with a given amount of initial private wealth.

Entrepreneurs can invest personal funds in their firm, and borrow from an outside lender, e.g., a bank. In each period the entrepreneur chooses the scale of investment and the fraction that is self-financed. The remaining fraction of firm capital is borrowed from the lender via a debt contract with an interest rate (or face value) that equates the lender’s expected payoff to the cost of funds. In the next period the project’s random return is realized and the entrepreneur chooses whether to repay the loan or default. If firm assets are not sufficient to repay the debt, the entrepreneur can use private assets to cover the difference. If instead default is chosen, firm assets are liquidated at a cost, their value is assigned to the lender, and the entrepreneur is excluded from the credit market and unable to operate a firm for several periods. The paper has four main results.

First, we decompose an entrepreneur’s default decision into three elements: the initial level of firm debt, the reduction in personal consumption from losing the firm, and the ratio of personal wealth to firm size. In static models, only the first effect is present and default occurs if and only if firm assets are less than debt. In contrast, in our dynamic model continuing to operate the firm has an option value. As a consequence, an entrepreneur may choose to inject additional personal funds to pay the firm’s debt to avoid default. The entrepreneur’s ability to bail out the firm depends on the ratio of personal wealth to firm size, which we refer to as the “wealth-asset ratio effect.” Willingness to bail out the firm depends on a “consumption loss effect,” which is the entrepreneur’s permanent loss in consumption from shutting down the firm. We show that these two effects are monotone in risk aversion, but move in opposite directions. More risk averse entrepreneurs run smaller firms in comparison to their wealth, i.e., the wealth-asset ratio is large, and they are better able to inject personal funds into their firms. Willingness to avoid default is smaller, however, since losing a small firm results in a small consumption loss. Quantitative analysis is necessary to

¹<http://www.forbes.com/sites/actiontrumpseverything/2012/04/12/are-you-risk-adverse-you-could-be-the-perfect-entrepreneur>

determine the outcome of these countervailing effects, which we provide in a calibrated model.

Second, we identify an existence problem that arises when firm scale is variable. Running an extremely (i.e., infinitely) leveraged firm can be profitable because if the failure rate is less than one, the entrepreneur's payoff may be infinite. The lender gets the entire project with probability close to one, and unless liquidation costs are very high, this is enough to cover the lender's cost of funds. Existence of a solution to the firm's problem must rule out such "gambling" by both parties. We do this by introducing a borrowing constraint, where the borrowing limit is set endogenously to a level at which the constraint does not bind locally, i.e., the Lagrange multiplier on the constraint is zero.² This problem would not arise if firm or investment size were fixed.

Third, we calibrate the model and show that with the endogenous borrowing constraint firms are more leveraged and entrepreneurs invest less wealth in the firm than in the data. This leads to larger firm size, but a smaller fraction of entrepreneurs because those who default are subsequently excluded from the credit market. We then use the model to estimate a borrowing constraint that better matches the data. The Lagrange multiplier is positive, indicating that entrepreneurs are credit constrained. This tighter constraint leads to a lower default rate, consistent with data. We find that up to a 50% reduction in default can occur, relative to the case where the constraint does not bind. We also permit risk aversion to differ, and use the model to estimate it. We find modest differences, with a median level of 1.6, in line with Mazzocco (2006) and parameter values used in the real business cycle literature. Thus, entrepreneurs do not seem to have very low risk aversion, rather they seem to use firm size, capital structure, default and consumption to manage risk.

Finally, we investigate the evolution of entrepreneur net worth, consumption, and firm assets over time, with tight and locally slack credit constraints. Cagetti and De Nardi (2006) document that wealth and entrepreneurship are closely connected.³ In our model entrepreneurs are willing to sacrifice significant consumption, using retained earnings to invest in their firms, in hope of achieving high future wealth. Although success may often elude them, the high potential gains of entrepreneurship, the downside protection from bankruptcy, and the ability to manage risk through firm size and capital structure explains entrepreneurs' willingness to try.

In the remainder, section 2 contains stylized facts. Section 3 specifies the model, an individual agent's problem, considers existence and characterizes default. Section 4 maps the model to U.S. data, discusses model fit, and analyzes the dynamic effects of risk aversion. Section 5 concludes.

²A local constraint means that if a much larger loan were possible, agents could gamble. This problem is distinct from standard Ponzi schemes, as it would occur even in a two period model.

³Depending on the definition of entrepreneur used, Cagetti and De Nardi (2006) Table 1 reports that U.S. entrepreneurs ranged from 7.6% of the population and hold 33% of total wealth to 16.7% of the population with 52.9% of total wealth, using 1989 Survey of Consumer Finances data.

2 Stylized Facts about Small Firms

We want the model to be consistent with several stylized facts, largely from the Survey of Small Business Finances (SSBF). This survey is administered by The Board of Governors of the Federal Reserve System and the U.S. Small Business Administration. Conducted in 1987, 1993, 1998, and 2003, each survey is a cross sectional stratified random sample of about 4000 non-farm, non-financial, non-real estate small businesses that represent about 5 million firms.⁴ The surveys contain information on the characteristics of small firms and their primary owner (e.g., owner age, industry, type of business organization), firm income statements and balance sheets, details on the use and source of financial services, and recent firm borrowing experience including trade credit and equity injections.

Fact 1: *Small firm returns are very risky.*

Table 1 provides summary statistics for real return on assets for small firms in the 1993 SSBF.⁵ The median return is about 9% and the mean is 30%. SSBF firms are noticeably risky, as the standard deviation indicates, with the high risk somewhat compensated by a high mean. About 12% of SSBF firms lost more than 20% of assets invested in the firm (debt plus equity), 7% lost more than 40%, and 4% lost more than 100%. Returns can also be substantial: about 21% exceeded 50%, 10% exceeded 100%, and 4% exceeded 200%. Skew is positive. The 95% confidence bands are computed using bootstrap sampling. For the median, the interquartile range is reported.

Table 1: Gross Real Firm Return Summary Statistics, 1993 SSBF

moment	median	mean	standard dev.	skewness	kurtosis
1993 SSBF	1.09	1.30	1.57	13	290
95% conf.	[1.08, 1.11]	[1.22, 1.38]	[0.95, 2.13]	[2.3, 17.3]	[29, 488]

Figure 1 shows the gross real return distribution. The distribution is tight around the median because variance is generated by some firms that do exceptionally well, which also generates the high kurtosis in table 1. Figure 1 shows that returns can be negative. To understand why this occurs, consider an entrepreneur who starts a bakery by borrowing \$5,000 from a bank for machines, uses trade credit to acquire perishable inputs worth \$2,000, and hires an employee promised a salary of \$3,000 at the end of the period. Suppose the business fails and earns \$0. If only the bank's

⁴All surveys are available at <http://www.federalreserve.gov>. See also Herranz, Krassa, and Villamil (2009).

⁵Only the 1993 survey has the interest expenses required to compute the return on assets (ROA). See section 6.2.

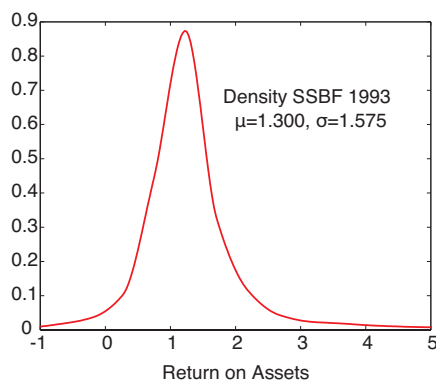


Figure 1: Empirical Gross Real Firm Return pdf, SSBF 1993

investment is recorded in the data, a total firm loss of \$10,000 on an initial investment of \$5,000 is reported, a gross return on assets of -1 .⁶

Fact 2: *Owners invest substantial personal net-worth in their firms.*

Table 2 reports the percentage of personal net-worth invested by entrepreneurs in their firm in the 1998 SSBF.⁷ The median amount of net-worth invested is 21%, but the data indicate a surprising lack of diversification for some entrepreneurs: 3% invest more than 80% of personal net-worth in their firm, 11% invest more than 60% and 25% invest more than 40%. This concentration of personal funds in a business is puzzling in view of the risky returns documented by table 1.

Table 2: Net-Worth Invested, 1998 SSBF

% net-worth invested	$\geq 20\%$	$\geq 40\%$	$\geq 60\%$	$\geq 80\%$	mean	median
% of entrepreneurs	52%	25%	11%	3%	27%	21%

Fact 3: *Most owners work at their firms.*

For incorporated firms, the percentage of primary owners who work at their firms is 79%, 89% and 87% in the respective 1993, 1998 and 2003 SSBFs. This fact compounds the risk return puzzle because if the firm fails, owners lose the funds invested and their jobs.

⁶The worker and trade credit supplier absorb losses of \$3,000 and \$2,000, respectively. This standard accounting issue does not matter for the quantitative part of our paper, except for a loan rate adjustment. Michelacci and Quadrini (2005) provide a model of wage dynamics where workers provide trade credit since wages are paid ex-post.

⁷Owner net-worth is listed only in the 1998 SSBF, consisting of personal net-worth plus home equity. We report the percentage of net-worth invested for incorporated firms with positive net-worth outside the firm, for firms with non-negative equity and assets of at least \$50,000. This lower bound on assets is the smallest number that did not generate numerical problems in our empirical analysis but left almost all of the sample intact.

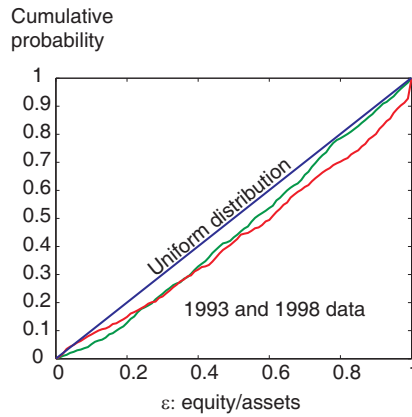


Figure 2: Cdfs of Equity/Assets for Firms with Positive Equity: 1993 and 1998 SSBF Data

Fact 4: *The average annual default rate on small business loans is 3.5-4.5%.*

Glennon and Nigro (2005) find the default rate on small business loans guaranteed by the Small Business Administration is 3.5% and Boissay and Gropp (2007), table 2.4, find a default rate on trade credit of 4.5% for small French firms (trade credit is a third of all firms' total liabilities in most OECD countries.)

Fact 5: *In 1993 and 1998, 15.7% and 21% of incorporated firms have negative equity, respectively.*

Negative equity means that firm liabilities exceed assets, making it likely that the firm needs additional funds, which could be the owner's personal funds or unpaid bills absorbed by creditors.

Fact 6: *The distribution of firm capital structure is uniform.*

Figure 2 shows that the cdfs of Equity/Assets in the 1993 and 1998 SSBF are approximately uniform. By definition, total assets consist of debt plus equity, thus Equity/Assets is a measure of firm capital structure. A uniform cdf indicates that all capital structures are equally likely. If all firms were identical, and firms chose their debt-equity ratios optimally, we would expect to see a peaked distribution clustered around the common optimal Equity/Assets ratio. The approximately uniform distributions observed in both 1993 and 1998 indicate that firms chose different debt-equity ratios, which suggests that firms may be heterogeneous.⁸

The facts show that small firm returns are risky. Owners invest significant personal net worth and are likely to work at the firm, which compounds the risk. Many firms have negative equity, yet the default rate is low. Why do firms forbear in the face of such poor performance? Put differently, why do these entrepreneurs not default on their loans? Finally, all debt-equity ratios are equally

⁸The data do not identify the type of heterogeneity.

likely, suggesting heterogeneity. We now construct a model that is consistent with these facts.

3 The Model

Consider an economy with $t = 0, 1, \dots$ time periods. We begin with the problem of an individual, with discount rate β and utility function over consumption $u(C)$, where C denotes period consumption. This individual has initial endowment w_0 and access to a constant returns to scale production technology. If the individual operates the technology, the firm produces output x per unit of assets invested \mathcal{A} . The firm's return is given by random variable X with cumulative distribution function $F(x)$ and probability density function $f(x)$, which is strictly positive on support $[\underline{x}, \bar{x}]$ with $\underline{x} \leq 0$, $\bar{x} > 0$, and X iid across periods. A negative realization means firm losses in a year exceed current assets, and the owner either uses personal funds to stay solvent or defaults.⁹ In all periods $t \geq 1$, the individual's net-worth w_t is derived from the return on investment in the firm and in an alternative investment opportunity with return r . Net-worth is known at the beginning of each period.

An entrepreneur is an individual who runs a firm by making an investment $\mathcal{A} > 0$.¹⁰ At any time t , the entrepreneur chooses the fraction of self-finance (equity) ϵ , and debt finance $1 - \epsilon$. Debt is provided by a risk-neutral competitive lender with an elastic supply of funds. The lender makes one-period loans.¹¹ The total amount of self-finance is therefore $\epsilon\mathcal{A}$ and the entrepreneur's opportunity cost of funds is $\epsilon\mathcal{A}(1 + r)$. For the total amount of funds borrowed, $(1 - \epsilon)\mathcal{A}$, the entrepreneur owes $\bar{v}\mathcal{A}$ next period. Thus, the loan rate is given by $r_L = \bar{v}/(1 - \epsilon) - 1$. The face value of debt \bar{v} , or equivalently r_L , is determined endogenously from the lenders' break even condition, given the risk free rate on the lenders' cost of funds r_f . In summary, there are two exogenous interest rates, r and r_f , and the endogenously determined loan rate r_L . In the model calibration, we will allow r to be larger than r_f . This reflects the fact that net worth w_t includes less liquid assets, such as home equity or retirement savings, that are more costly to use. In the calibration the endogenous loan rate r_L will exceed r , i.e., the cost of debt is higher than self-finance if there is no default. This cost advantage of self-finance is counterbalanced by the fact that in the case of default an entrepreneur with more debt-finance will lose less personal funds.

After production occurs the firm has assets $\mathcal{A}x$ and liabilities $\mathcal{A}\bar{v}$, and the entrepreneur chooses

⁹As we explain for Figure 1, $x < 0$ can occur if the firm has trade credit.

¹⁰An individual that does not wish to run a firm sets $\mathcal{A} = 0$.

¹¹We consider a composite lender that supplies all liabilities (bank loans, trade credit and other liabilities) and can infer borrower risk aversion. The average maturity on loans to small firms is less than one year in the Federal Reserve's *Survey of Terms of Business Lending* (these firms lack audited financial statements, payment or profit histories, or verifiable contracts with workers, suppliers or customers).

whether to repay loan $\mathcal{A}\bar{v}$ or default.¹² When default occurs, bankruptcy follows immediately and is described by two parameters, δ and T . A court determines the total value of firm assets, transferring a fraction $1 - \delta$ to the lender, where δ is a deadweight bankruptcy loss (e.g., firm assets are sold at a loss).¹³ Since the firm is incorporated, the entrepreneur is protected by limited liability and only firm assets can be seized. If bankruptcy occurs, the entrepreneur does not have access to the firm's returns for T periods, which has two interpretations. First, corresponding to Chapter 7 in the U.S. Bankruptcy Code, the firm may be liquidated. Because bankruptcy remains on a credit record for a period of time, creditors and customers would be unwilling to do business with the entrepreneur during this period. Second, corresponding to Chapter 11, the firm may continue to operate, but is owned by the debt-holders who make investments and receive payments, or shut it down. After T periods the credit record is clean, and the entrepreneur can either restart a new firm or regain control of the original firm, in Chapter 7 or 11 respectively.

The timing of events for incorporated firms is as follows:

1. Beginning of period t (*ex-ante*) entrepreneur net-worth is w . There are two cases:
 - (a) *The entrepreneur did not declare bankruptcy in any of the previous T periods:* Choose consumption C , firm assets \mathcal{A} , self-finance ϵ (debt is $1 - \epsilon$), and amount $\mathcal{A}\bar{v}$ to repay, subject to the lender receiving at least ex-ante expected payoff $(1 - \epsilon)\mathcal{A}(1 + r_f)$.
 - (b) *The entrepreneur declared bankruptcy k periods ago.* The owner cannot operate the firm for the next $T - k$ periods. Hence, only current consumption is chosen.
2. At the end of period t (*ex-post*) the firm's return on assets, x , is realized. Total end-of-period firm assets are $\mathcal{A}x$. The entrepreneur must decide whether or not to default. If
 - (a) *Default:* Only firm assets are seized. The entrepreneur is left with personal net-worth $(1 + r)(w - \epsilon\mathcal{A} - C)$, invested at outside interest rate r .
 - (b) *No Default:* Entrepreneur net-worth is $\mathcal{A}(x - \bar{v}) + (1 + r)(w - \epsilon\mathcal{A} - C)$, which includes both net-equity in the firm and the return on personal assets.

¹²A firm may default if it is unable to repay $\mathcal{A}\bar{v}$ (firm plus personal assets are less than \mathcal{A}) or is unwilling to repay. The entrepreneur can "bail out the firm" with personal assets to forestall bankruptcy, but cannot be forced to do so. The owner's personal credit history affects business loans, causing a credit interruption. Mester (1997) p. 7 finds that in small business loan scoring models, "the owner's credit history was more predictive than net worth or profitability of the business" and "owners' and businesses' finances are often commingled."

¹³Athreya (2004) considers an additional cost, stigma, which would lower the default rate.

3.1 The Problem of an Individual Entrepreneur

Consider the optimization problem of an individual entrepreneur, with a given coefficient of risk aversion ρ and net-worth w at the beginning of the period. We state the problem recursively. Our initial goal is to determine the structure of the value function. If bankruptcy occurred in the previous T periods, then the state is given by (B, k, w) where k is the number of periods since default and B denotes bankruptcy. Otherwise, the state is given by (S, w) , and S denotes solvency. Denote the value functions by $V_{B,k}(w)$ and $V_S(w)$, respectively. After T periods the firm can restart, thus $V_{B,T}(w) = V_S(w)$. Let \mathfrak{B} denote the set of asset return realizations x for which bankruptcy occurs, with complement \mathfrak{B}^c . We specify a problem for each default state.

If the firm did not default in the previous T periods, the individual solves:

Problem 1

$$V_S(w) = \max_{C \geq 0, \mathcal{A} \geq 0, 0 \leq \epsilon \leq 1, \bar{v}} \left\{ u(C) + \beta \left[\int_{\mathfrak{B}} V_{B,1}((1+r)(w - \epsilon\mathcal{A} - C)) dF(x) + \int_{\mathfrak{B}^c} V_S(\mathcal{A}(x - \bar{v}) + (1+r)(w - \epsilon\mathcal{A} - C)) dF(x) \right] \right\}$$

Subject to:

$$\int_{\underline{x}}^{\bar{x}^*} \min\{(1-\delta)x, x\} dF(x) + \int_{x^*}^{\bar{x}} \bar{v} dF(x) \geq (1-\epsilon)(1+r_f) \quad (1)$$

$$x \in \mathfrak{B} \text{ if and only if } V_{B,1}((1+r)(w - \epsilon\mathcal{A} - C)) > V_S(\mathcal{A}(x - \bar{v}) + (1+r)(w - \epsilon\mathcal{A} - C)) \quad (2)$$

$$(1-\epsilon)\mathcal{A} \leq bw \quad (3)$$

The objective is the utility of current consumption plus the discounted continuation value of end of period net-worth.¹⁴ Constraint (1) ensures that the lender is willing to supply funds. The right-hand-side indicates the fraction, $1 - \epsilon$, of funds the lender invests in the firm must earn at least reservation return $1 + r_f$. The left-hand side is the lender's expected return from the loan per unit of assets \mathcal{A} in default and solvency. In default, if $x > 0$ and there is a positive amount to seize deadweight loss δ arises in transferring assets, and if $x < 0$ the lender absorbs the loss. The second term is the fixed debt repayment when the firm is solvent. Constraint (2) specifies ex-post

¹⁴Ex ante $0 \leq \epsilon \leq 1$, but ex-post negative equity may occur. This distinction arises because the non-negativity constraint on equity only applies ex ante. Ex-post, if project realization x is low, assets are low and end-of-period equity will be negative due to the accounting identity: assets = debt + equity.

optimality of the default decision: An entrepreneur defaults if and only if the expected continuation payoff after default exceeds that from solvency.¹⁵ Note that if $\mathcal{A} = 0$, i.e., the entrepreneur does not run a firm, then inequality (2) is never satisfied and the bankruptcy set is empty. Borrowing constraint (3) may limit business loans to fraction b of entrepreneur net-worth. We discuss this constraint in the next section.

Now consider the problem of a firm that defaulted $k \leq T$ periods ago. After T periods the firm can operate again, thus $V_{B,T}(\cdot) = V_S(\cdot)$. Let w' denote net-worth next period:

Problem 2
$$V_{B,k}(w) = \max_{w' \geq 0} \left\{ u \left(w - \frac{w'}{1+r} \right) + \beta V_{B,k+1}(w') \right\}$$

The objective is expected ex-ante utility with budget constraint $C(1+r) + w' \leq w(1+r)$ substituted in, satisfied at equality. If default occurred the entrepreneur cannot run the firm for T periods.

Theorem 1 uses the fact that CRRA utility is scalable in wealth to determine the structure of the value function, which allows us to restate Problems 1 and 2 as a one-dimensional fixed point problem. The proof is in Appendix B.

Theorem 1 *Suppose that the entrepreneur has constant relative risk aversion:*

1. *Let $v_S = V_S(1)$ and $v_{B,k} = V_{B,k}(1)$. Then $V_S(w) = w^{1-\rho}v_S$ and $V_{B,k}(w) = w^{1-\rho}v_{B,k}$.*
2. *Let c, A, ϵ, \bar{v} be the optimal choices of consumption, production scale, equity structure and debt face value starting with initial wealth $w = 1$ in a period. Then $C = cw$, $\mathcal{A} = Aw$, ϵ , and \bar{v} are the optimal values when starting with wealth w .*

In view of statement 1 of Theorem 1 it is sufficient to determine v_S and $v_{B,k}$ to determine the entire value function. Using Problem 2, it is straightforward to compute $v_{B,k}$ as a function of v_S . In Problem 1 we only need $v_{B,1}$, the continuation utility given that default was just announced, and v_S . To simplify notation, write v_B for $v_{B,1}$. Thus, it remains to specify the recursive optimization problem when starting with an initial wealth of $w = 1$. In view of the second statement of Theorem 1 we can replace C and \mathcal{A} by c and A , the values obtained when starting with one unit of wealth.

Lemma 1 and Lemma 2 in Appendix B prove that the investor's constraint binds and bankruptcy set \mathfrak{B} is a lower interval with cutoff x^* . Thus, the problem can be rewritten as follows, where all endogenous variables are now expressed as a fraction of net-worth w :

¹⁵Bailing out the firm with personal funds means that the entrepreneur continues to operate the firm even if $x < \bar{v}$. In a one period model (instead of the dynamic model) both $V_{B,1}$ and V_S would be the identity mapping, and (2) would reduce to $x \in \mathfrak{B}$ if and only if $(1+r)(w - \epsilon\mathcal{A} - C) > \mathcal{A}(x - \bar{v}) + (1+r)(w - \epsilon\mathcal{A} - C)$, which implies $x \in \mathfrak{B}$ if and only if $x < \bar{v}$ (bankruptcy occurs if and only if the return is less than debt plus interest).

Problem 3

$$v_S = \max_{c \geq 0, A \geq 0, 0 \leq \epsilon \leq 1, \bar{v}} \left\{ \frac{c^{1-\rho}}{1-\rho} + \beta v_B \int_{\underline{x}}^{x^*} [(1+r)(1-\epsilon A - c)]^{1-\rho} dF(x) \right. \\ \left. + \beta v_S \int_{x^*}^{\bar{x}} [A(x - \bar{v}) + (1+r)(1-\epsilon A - c)]^{1-\rho} dF(x) \right\}$$

Subject to: (1) and

$$x^* = \max \left\{ \bar{v} - \left[1 - \left(\frac{v_B}{v_S} \right)^{\frac{1}{1-\rho}} \right] \frac{(1+r)(1-\epsilon A - c)}{A}, \underline{x} \right\} \quad (4)$$

$$c + \epsilon A \leq 1 \quad (5)$$

$$(1 - \epsilon)A \leq b \quad (6)$$

The objective is the utility of current consumption and the discounted value of end of period networth, with default set $[\underline{x}, x^*]$ and solvency set $[x^*, \bar{x}]$.¹⁶ Constraint (1) is lender individual rationality, which binds by Lemma 1 in Appendix B. Constraint (4) is the optimal default cutoff and follows from Lemma 2. The default cutoff equation is only valid if $A > 0$; if $A = 0$ the individual does not operate a firm. (5) ensures feasibility and (6) is the borrowing constraint.

3.2 Existence, Uniqueness and the Borrowing Constraint

The individual's problem is stated for a given risk aversion parameter ρ . In this section we show that solutions to Problem 3 exist unless risk aversion is below a cutoff value, and we explain the borrowing constraint's role in ensuring boundedness of the constraint set. We consider risk aversion and the borrowing constraint separately in order to better understand their distinct roles. We first state the main existence result. The proofs are in Appendix B.

Theorem 2 *Suppose $r \geq 0$. Then there exists $\underline{\rho} < 1$ such that Problem 3 has a solution for all $\rho \geq \underline{\rho}$.*

In the proof, $\Lambda(v_S)$ is expected utility given continuation value v_S . The proof uses the intermediate value Theorem to show that there exists a fixed point $\Lambda(v_S^*) = v_S^*$. In the computational part we use Newton's method to determine the fixed point; iteratively applying Λ does not work since the derivative of $\Lambda(\cdot)$ near v_S^* is typically greater than 1, and hence Λ is not a contraction mapping.¹⁷

¹⁶Since our distributions are non-atomic it does not matter whether we assign x^* to the bankruptcy or solvency set.

¹⁷The argument that the plan generated by the recursive problem solves the infinite horizon problem is standard and follows from Theorem 9.2 of Stokey and Lucas (1989).

3.2.1 Non-existence for Low Risk Aversion

No solutions exist for $\rho < \underline{\rho}$ because the product of β and expected gross return $1 + r$ exceeds 1, and an individual with sufficiently low risk aversion would defer consumption indefinitely. This is obvious for a risk-neutral individual and by continuity extends to moderately risk averse entrepreneurs. In order to better understand the model, we compute the lower-cutoff for existence $\underline{\rho}$ in the simple case where the investment technology is deterministic and there is no outside finance. We make the latter assumption to show that non-existence depends on the scalability of the project but not on an entrepreneur's ability to choose the firm's financial structure.

Let i denote investment, R denote its return, e denote endowments, and $t = 0, 1, \dots$ denote time. The entrepreneur solves the following optimization problem:

$$\max_{c_t, i_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\rho}}{1-\rho} \quad \text{subject to} \quad c_t + i_{t+1} = e_t + Ri_t \quad (7)$$

The first order conditions yield: $c_{t+1} = (R\beta)^{1/\rho} c_t$, or $c_t = (R\beta)^{t/\rho} c_0$. It is easy to check that if this solution for c_t is substituted into the objective, the sum is finite if $\underline{\rho} > 1 - \frac{\ln(1/\beta)}{\ln(R)}$. We can get existence for lower ρ by reducing β , but this is not economically plausible (i.e., entrepreneurs become too impatient to make sizeable investments).

The failure of existence for very low levels of risk aversion depends crucially on flexible investment size. If project size is fixed at \bar{A} , then utility is always bounded because consumption each period can be at most $\bar{A}R$.

3.2.2 Non-Existence without the Borrowing Constraint

We now show that absent borrowing constraint (6) solutions to Problem 3 may not exist, because a firm's ability to choose scale and capital structure results in a "gambling" problem, where entrepreneurs would run extremely (i.e., infinitely) leveraged firms.

Assume that investment project X , bankruptcy cost δ , and the risk-free rate r_f satisfy

$$E[X] > 1 + r_f - \delta. \quad (8)$$

In table 1 in section 2, $E[X] = 1.3$. The risk-free rate in the U.S. is below 5%, thus (8) is satisfied even if bankruptcy losses are a quarter of firm assets. Boyd and Smith (1994) find δ is 10%. Bris, Welch, and Zhu (2006) estimate costs of 0-20% of assets, which implies that (8) holds.

To see that no solution exists to the entrepreneur's optimization problem if (8) is holds, choose ϵ_n, A_n such that $\epsilon_n A_n = 1/n$ and $A_n \rightarrow \infty$, i.e., the entrepreneur increases firm size to infinity, but

equity in the firm converges to zero at the same time. Let \bar{v}_n be the face value of the debt that satisfies borrowing constraint (6) from Problem 3. Then (1) and (8) imply $\hat{v} = \limsup_{n \rightarrow \infty} \bar{v}_n < \bar{x}$, else the left-hand side of (1) is strictly larger than the right-hand side. As a consequence, (4) implies that $x^* \rightarrow \hat{v}$ since $(1+r)(1 - \epsilon_n A_n - c)/A_n = (1+r)(1 - (1/n) - c)/A_n \rightarrow 0$, in other words the probability of default is strictly bounded away from 1. Thus, as $n \rightarrow \infty$ the entrepreneur's payoff converges to infinity since

$$\lim_{A_n \rightarrow \infty} \beta v_S \int_{\hat{v}}^{\bar{x}} \left[A_n(x - \bar{v}) + (1+r)(1 - \epsilon_n A_n - c) \right]^{1-\rho} dF(x) = \infty \quad (9)$$

Intuitively, the entrepreneur runs an infinitely sized firm, but defaults with probability below 1. As a consequence, he receives an infinite payoff with positive probability. At the same time the investor gets all of the project return most of the time, which is sufficient to cover the lender's cost of funds. In this case, the entrepreneur's optimization problem has no solution. The borrowing constraint remedies this. The problem is not related to existence problems caused by Ponzi schemes. Instead, it is generated by the entrepreneur's ability to choose both firm size and capital structure, which allows the entrepreneur to run an extremely leveraged firm.¹⁸

3.2.3 Locally Slack Borrowing Constraints

Borrowing constraints are essential in dynamic models with incomplete markets, but a key question is how to endogenize the constraint. Alvarez and Jermann (2000) establish a second welfare theorem for an economy with limited commitment, by introducing "not-too-tight" constraints on borrowing. These constraints, which are taken as given by agents in the competitive equilibrium, ensure that all market participants are willing to pay back their debt in all states, while simultaneously permitting as much risk sharing as possible.¹⁹ Chatterjee, Corbae, Nakajima, and Rios-Rull (2007), Livshits, MacGee, and Tertilt (2007), and Arellano (2008) consider models with incomplete markets where, unlike Alvarez and Jermann (2000), default occurs in equilibrium. For example, in Chatterjee, Corbae, Nakajima, and Rios-Rull (2007) borrowing is possible until the household's level of debt reaches a point at which the probability of repayment becomes zero. At this point the price of debt becomes zero, and no lender will provide additional funds. This approach does not work in our framework because of the non-existence problem: both the lender and the borrower are better off when the firm gambles by using extreme levels of leverage. Unlike in

¹⁸Our effect differs from Vereshchagina and Hopenhayn (2009), where entrepreneurs choose risky projects to generate lotteries that convexify the objective function.

¹⁹Martins-da Rocha and Vailakis (2013) introduce a refinement that results in a unique not-too-tight constraint.

Chatterjee, Corbae, Nakajima, and Rios-Rull (2007), the repayment to the lender does not go to zero in our model even if the loan size goes to infinity.

To determine an analog of a not-too-tight constraint for our model we allow borrowing until gambling becomes an issue. Formally, this is achieved by defining a locally slack constraint. We use Problem 3 and relax b until the Lagrange multiplier on constraint (6) is zero. At this point entrepreneurs are locally unconstrained, i.e., they would not want to increase their level of debt by any small amount. However, the entrepreneur is constrained from choosing a much larger loan that would lead to very high leverage and gambling. In the quantitative analysis we will investigate whether our locally slack constraint or a tighter constraint (lower b) better fits the data.

3.3 Characterization of Default

Knowing the shape of the value function, we can now derive a formula that links firm and owner characteristics to the default decision. In Problem 3, constraint (4) gives default cutoff x^* . We decompose x^* into three distinct effects that we analyze individually. Let c_B and c_S be the constant consumption over time that would result in a utility of v_B or v_S , respectively. It follows that

$$\frac{c_B}{c_S} = \left(\frac{v_B}{v_S} \right)^{\frac{1}{1-\rho}}.$$

Assuming that default occurs with positive probability, i.e., $x^* \geq \underline{x}$, bankruptcy constraint (4) is equivalent to

$$x^* = \bar{v} - \left[\frac{c_S - c_B}{c_S} \right] \left[\frac{(1+r)(1-\epsilon A - c)}{A} \right].$$

The default cutoff terms are: ex ante firm debt \bar{v} , the consumption loss from firm bankruptcy, and the entrepreneur's personal net wealth outside the firm over firm scale, which determines the entrepreneur's ability to inject more personal funds into the firm. In words,

$$x^* = \text{ex-ante debt} - \text{consumption loss} \times \text{wealth-to-firm-scale ratio.} \quad (10)$$

Consider the three forces that determine default cutoff x^* :

Ex-ante debt: In static models, agents default if their assets x are less than debt \bar{v} , and hence all firms with negative equity default, cf., Townsend (1979) and Gale and Hellwig (1985).

Consumption loss: This term measures the percentage decline in consumption from losing the firm, where c_S and c_B are the constant consumption streams that yield the same utility as the entrepreneur's actual consumption in solvency and bankruptcy states, respectively.

Wealth-to-firm-scale ratio: In order to prevent firm bankruptcy, an entrepreneur can inject personal assets held outside the firm to cover firm debt \bar{v} . This is easier to do if the firm is small relative to the entrepreneur's net-worth.

If only the first term were present, all firms with negative equity would default, and the fraction of firms with negative equity and the default rate would coincide. Facts 4 and 5 in Section 2 show there is a big gap between the two numbers, which indicates the importance of the last two terms of (10). We derive theoretical results to provide insight into the relationship between risk aversion and the last two terms. Theorem 3 states conditions under which the consumption loss is decreasing in risk-aversion. The intuition is that more risk-averse agents invest a smaller fraction of their assets in the firm and, as a consequence, the difference between default and non-default utility is small. Theorem 4 shows that the wealth-to-firm-scale ratio is decreasing as long as risk-aversion and default are not too large. Intuitively, more risk averse individuals run smaller firms and hence the ratio of wealth outside the firm to firm assets is large.

Theorem 3

1. *Suppose that $(1 + r)\beta = 1$ and $T = \infty$. Then the consumption loss is decreasing in ρ .*
2. *For given ρ , there exists a $\bar{T} > 0$ and such that the consumption loss is decreasing in ρ for any $T \geq \bar{T}$, when $\beta(1 + r)$ is in a neighborhood of 1.*

We next show that ϵ and A are both monotonically decreasing in ρ , i.e., a more risk averse individual will choose a higher debt-equity ratio and a lower project scale.

Theorem 4 *Suppose that $(1 + r)\beta = 1$, $T = \infty$ and the borrowing constraint is locally slack. Let density f of return distribution F be continuous, and assume that $F(x) = 0$ implies $f(x) = 0$. Then ϵ and A are decreasing in ρ for all sufficiently low levels of risk aversion and for sufficiently small default probabilities.*

Theorem 4 indicates two major channels through which an entrepreneur can lower risk. First, the entrepreneur can run a project at a smaller scale. Ceteris paribus, this lowers the default probability since the wealth-to-firm-scale ratio is increased. Second, a more risk averse entrepreneur would wish to increase the project's leverage. In this case, leverage is not used to increase project scale, but rather to reduce the amount of entrepreneur funds tied up in the firm. As a consequence, more entrepreneur funds can be invested outside the firm at riskless rate r , providing a cushion if default occurs. The effect on the default probability from increased leverage is ambiguous: Ceteris

paribus lowering ϵ raises the wealth-to-firm-scale ratio, which reduces default. However, reducing ϵ means the firm borrows more, increasing ex-ante debt level \bar{v} (see constraint (1)), and raising the default probability. We will see in the calibrated model that the first effect dominates the second, and default is generally lower as ρ decreases. We now calibrate the model.

4 Mapping the Model to U.S. Data

Table 3: Exogenous Parameters

Parameter	Interpretation	Value	Comment/ Observations
r_f	lender opportunity cost	1.2%	real rate, 6 mo T-Bill, 1992-2006
r	entrepreneur opportunity cost	4.5%	real rate, 30 year mortgage, 1992-2006
β	discount factor	0.97	determined from r and r_f
T	default exclusion period	11	U.S. credit record
δ	default deadweight loss	0.10	Boyd-Smith (1994)
$f(x)$	pdf of firm returns		SSBF 1993 (Appendix D)

We use U.S. data to assign values to five model parameters and the distribution of firm returns in table 3. We identify r_f , the lender’s opportunity cost of short-term funds, with the average real return on 6 month Treasury bills between 1992 and 2006.²⁰ The interest rate charged by the lender will be strictly higher than r_f because of bankruptcy costs. We identify the owner’s opportunity cost of funds r with the real rate on 30 year mortgages over the period; the cost of using home equity to finance a business loan will also be strictly higher. $\beta = 0.97$ is approximated by $1/(1+0.5r_f+0.5r)$, with r and r_f weighed equally (firm risk cannot be diversified since a portfolio of small firms does not exist). The bankruptcy exclusion parameter is $T = 11$, because in the U.S. after 10 years past default is removed from a credit record. The bankruptcy deadweight loss is $\delta = 0.1$, as in Boyd and Smith (1994) and the midpoint of 0 – 20% of assets in Bris, Welch, and Zhu (2006). Firm return distribution $f(x)$ is computed from SSBF data on incorporated firms, see Fact 1 in section 2 and section 6.2. The remaining parameters are jointly calibrated by choosing μ , σ and possibly b to minimize the distance between model predictions and SSBF data.

4.1 Heterogeneous risk aversion

In order to match the SSBF data we must construct distributions, which in turn requires us to model a source of underlying heterogeneity. Because willingness to bear risk is at the heart of

²⁰We use monthly data for T-Bill rates and deduct for each month the CPI reported by the BLS.

entrepreneurship, we extend the model specified for an individual agent to a setting where agents may be heterogeneous with respect to risk aversion parameter ρ . We first construct distributions in the SSBF, and then construct the counterpart implied by the model.

Empirical Distributions: We focus on three empirical distributions derived from SSBF data that are important for small firms: firm assets, personal net-worth invested in the firm, and the ratio of equity over assets. Since in our model all quantities are normalized by the entrepreneur's net-worth, we define them as follows:

$$\begin{aligned} \text{Normalized net worth: } w &= \frac{\text{owners' share * equity}}{\text{net-worth outside the firm} + \text{owners' share * equity}} \\ \text{Normalized assets: } a &= \frac{\text{owners' share * asset}}{\text{net-worth outside the firm}} \\ \text{Equity-Asset ratio: } e &= \frac{\text{owners' equity in firm}}{\text{firm assets}} \end{aligned}$$

We denote the cdfs of the empirical distributions by $W^e(w)$, $W^e(a)$, and $E^e(e)$, respectively. We construct $W^e(w)$ only for firms with non-negative equity, to avoid possible division by zero.

Distributions implied by the model: Let $\phi_{\mu,\sigma}(\rho)$ be a normal distribution over risk aversion, and $\Phi_{\mu,\sigma}(\rho)$ its cdf. Given firm return pdf $f(x)$ the following cdfs are predicted by the model:

Cdf of Net-Worth. After x is realized, firm assets are $A(\rho)x$ and debt is $A(\rho)\bar{v}$. Equity in the firm is $A(\rho)(x - \bar{v}(\rho))$, which is non negative if $x \geq \bar{v}(\rho)$, while the entrepreneur's net-worth outside the firm is $(1+r)(1-c(\rho) - \epsilon(\rho)A(\rho))$. The fraction of total net-worth invested is therefore

$$w = \frac{A(\rho)(x - \bar{v}(\rho))}{A(\rho)(x - \bar{v}(\rho)) + (1+r)(1-c(\rho) - \epsilon(\rho)A(\rho))}. \quad (11)$$

Solving (11) gives $x = x(w, \rho)$. Since w is strictly increasing in x , the entrepreneur's net-worth invested in the firm is less than or equal to w if and only if $x \leq x(w, \rho)$. Finally, since we only use firms with positive equity to compute net-worth invested for the empirical distribution, we must do the same for the model predicted distribution, i.e., we restrict attention to firms with return realizations $x \geq \bar{v}(\rho)$. For firms with positive equity, the model-predicted cdf of net worth invested in the firm is therefore given by²¹

$$W_{\mu,\sigma}^m(w) = \frac{\int_{\bar{v}(\underline{\rho})}^{x(w,\underline{\rho})} f(x)\Phi_{\mu,\sigma}(\underline{\rho}) dx + \int_{\underline{\rho}}^{\infty} \int_{\bar{v}(\rho)}^{x(w,\rho)} f(x)\phi_{\mu,\sigma}(\rho) dx d\rho}{\int_{\bar{v}(\underline{\rho})}^{\infty} f(x)\Phi_{\mu,\sigma}(\underline{\rho}) dx + \int_{\underline{\rho}}^{\infty} \int_{\bar{v}(\rho)}^{\infty} f(x)\phi_{\mu,\sigma}(\rho) dx d\rho}. \quad (12)$$

²¹The denominator is the probability that the entrepreneur has positive equity, where ρ is the lowest parameter for which a model solution exists. For all $\rho < \underline{\rho}$ we assign the model solution for $\underline{\rho}$. See section 4.3.

Cdf of Equity/Assets. After project return x is realized, the fraction of equity is given by

$$e = \frac{A(\rho)(x - \bar{v}(\rho))}{A(\rho)x} \quad (13)$$

Let $x = x(e, \rho)$ solve (13). Then for firms with positive equity, the cdf of equity/assets is

$$E_{\mu, \sigma}^m(e) = \frac{\int_{\bar{v}(\rho)}^{x(e, \rho)} f(x) \Phi_{\mu, \sigma}(\rho) dx + \int_{\rho}^{\infty} \int_{\bar{v}(\rho)}^{x(e, \rho)} f(x) \phi_{\mu, \sigma}(\rho) dx d\rho}{\int_{\bar{v}(\rho)}^{\infty} f(x) \Phi_{\mu, \sigma}(\rho) dx + \int_{\rho}^{\infty} \int_{\bar{v}(\rho)}^{\infty} f(x) \phi_{\mu, \sigma}(\rho) dx d\rho}. \quad (14)$$

Cdf of End of Period Assets. The current realization of end of period assets as a fraction of net-worth outside the firm is:

$$\alpha = \frac{A(\rho)x}{(1+r)(1-c(\rho) - \epsilon(\rho)A(\rho))} \quad (15)$$

Let $x = x(\alpha, \rho)$ solve (15). Then the cdf of end of period assets is

$$A_{\mu, \sigma}^m(\alpha) = \int_{\underline{x}}^{x(\alpha, \rho)} f(x) \Phi_{\mu, \sigma}(\rho) dx + \int_{\rho}^{\infty} \int_{\underline{x}}^{x(\alpha, \rho)} f(x) \phi_{\mu, \sigma}(\rho) dx d\rho. \quad (16)$$

4.2 Model Calibration

Given the values for the exogenous model parameters in Table 3, we use the model to compute the remaining parameters in the two possible borrowing constraint cases, slack and tight:

Locally slack constraint: In this case only μ and σ must be determined. Choose μ, σ to minimize the supnorm distance between the cdf implied by the model and the cdf from the SSBF data:

$$\min_{\mu, \sigma \geq 0} \|W_{\mu, \sigma}^m(w) - W^e(w)\|_{\infty} + (0.431 - \alpha_{\mu, \sigma})^+ + (\alpha_{\mu, \sigma} - 0.519)^+ \quad (17)$$

The cdf of net-worth invested implied by the model, $W_{\mu, \sigma}^m(w)$, is given by (12). The supremum norm $\|\cdot\|_{\infty}$ is taken over all non-negative fractions of net-worth.²² The second and third terms impose penalties only for asset values outside the 95% confidence interval for firm assets, which Herranz, Krasa, and Villamil (2009) find is [43.1, 51.9]. Since we exclude firms with negative equity when determining W^e , net-worth invested is between 0% and 100%, but assets are unbounded.²³

Tight constraint: In this case, solve problem (17) over b, μ, σ .

²²To compute the supremum norm we evaluate $|W_{\mu, \sigma}^m(w) - W^e(w)|$ at 1,000 equi-distant points between 0 and 1, and take the maximum.

²³For example, 5% of firms had assets over ownership share that exceeded owner net-worth by 500%.

Table 4: Calibrated Parameters and Fit

Parameter	Interpretation	Model with Endog. b	Model with Fixed b
$b\%$	borrowing constraint: loan $\leq bw$	NA	21.5
μ	median of distribution of risk aversion	1.56	1.55
σ	standard deviation of distribution of risk aversion	1.50	0.83
fit	Distance to fitted cdf & assets in (17)	0.145	0.042

4.3 Performance of the Model

Table 4 reports parameter values for both cases, first where b varies endogenously such that the borrowing constraint is locally slack, and second where b is determined by (17) for all entrepreneurs. In both cases the table shows that the median entrepreneur risk aversion is between 1.5 and 1.56. To put the estimates in perspective, Mazzocco (2006) uses the Consumer Expenditure Survey to estimate a median coefficient of risk aversion of 1.7 for men. While one may expect entrepreneurs to be somewhat less risk averse than the general population, the model indicates that matching the data does not require very low levels of risk aversion. We find that σ is 1.50 or 0.83 for the two cases, where non-zero values indicate agent heterogeneity.²⁴ Table 4 shows the model with a tight constraint has a better fit than the slack constraint, 0.042 versus 0.145.

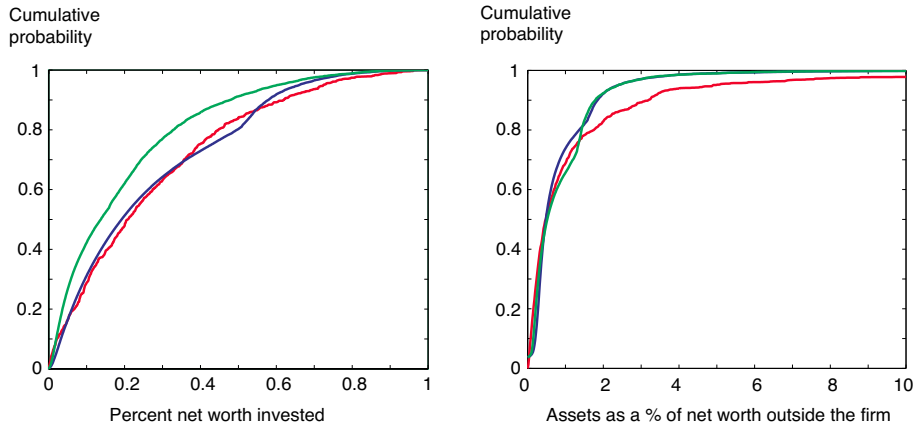


Figure 3: 1998 SSBF data and model with slack & tight constraints: net-worth & assets cdfs

²⁴Mazzocco (2006) does not estimate the distribution of risk aversion, so his estimate of the standard deviation of 0.96 is not directly comparable to ours. The calibrated parameters do not vary significantly with legal parameters δ and T . The insensitivity to changes in δ is due to the low equilibrium default rate. The best model fit is obtained at a value of $T = 13$. Thus, if we calibrate T instead of choosing it to be consistent with U.S. institutions, the numbers for the calibrated parameters and model results do not change significantly.

The left panel of figure 3 shows the cdf for net-worth invested in the SSBF data (red), the model with tight borrowing constraint b (blue), and the locally slack constraint (green).²⁵ The model with a slack constraint does not match the level that entrepreneurs invest in their firm. Intuitively, a slack constraint makes it possible for an entrepreneur to reduce risk by putting less personal funds at stake. This generates the higher level of default in table 5. The tighter credit constraint forces entrepreneurs to use more of their own money and this reduces default.²⁶ The right panel shows asset levels in the SSBF data and the predictions of the two models. Both models do not match the assets of a few large firms for reasons explained below, but compared to each other the model results are similar. Table 5 shows the overall point predictions largely fall within the data ranges.

Our model is quantitatively plausible along a number of dimensions. As discussed above, the left panel of figure 3 shows the fraction of net-worth an owner invests in the firm. Since we fit to this empirical cdf one would expect to see a match, but the constrained model does a very good job replicating the facts in table 2 in section 2. Owners invest substantial personal net-worth in their firms: the median is 21% and the mean is 27%. The right panel of figure 4 compares the predicted cdfs of firm assets to the data. The match is also good, but both models miss a few large firms.²⁷ The model predicted median asset levels of 48.1% and 51.9% in table 5 are within the 95% confidence interval of [43.1, 51.9].

Figure 4 compares the constrained model prediction (blue line) for firm capital structure to the cdfs for 1993 and 1998 SSBF data (red and green lines). The left panel shows that the model misses somewhat equity/assets. This occurs because no model solutions exist below $\underline{\rho}$. At $\underline{\rho} = 0.74$ the associated value of \bar{v} is 0.335. At median return level $\hat{x} = 1.094$, this gives an ex post value of equity/assets of $(\hat{x} - \bar{v})/\hat{x} = 0.7$, which is where the kink in the left panel occurs. If the cdf of ϵ is computed conditional on $\epsilon < 0.7$, the model does an excellent job of replicating the empirical distribution of equity/assets among firms – see the right panel. By definition total assets are debt plus equity, thus equity/assets is a measure of firm capital structure. The approximately uniform cdf indicates that all capital structures are equally likely, which is consistent with agent heterogeneity if individual firm capital structure is optimal.

²⁵Only the 1998 SSBF has owner net worth, personal net-worth plus home equity. The data cdf for net-worth invested is for firms with positive net-worth outside the firm, non-negative equity, and at least \$50,000 in assets.

²⁶Glover and Short (2011) also provide evidence that a tight credit constraint may apply for incorporated firms. They find that incorporated firms pay an interest rate premium compared to unincorporated firms, which they argue is a consequence of a “contraction in the supply of credit.” In other words, the credit constraint binds.

²⁷One reason the model predicts fewer very large firms is that solutions do not exist below $\underline{\rho} = 0.74$. At $\underline{\rho}$, the ex-ante level of ϵ and A are 0.720 and 0.766, respectively. Thus, end of period net-worth outside the firm, $(1 - \epsilon A - c)(1 + r)$ is about 0.470. Using median return $\hat{x} = 1.094$ from table 1, the ex-post level of assets as a fraction of net-worth for risk aversion level $\underline{\rho}$ is $A\hat{x}/(1 - \epsilon A - c)(1 + r) = 1.786$. In the figure, this is where the model predicted curves move away from the (red) data.

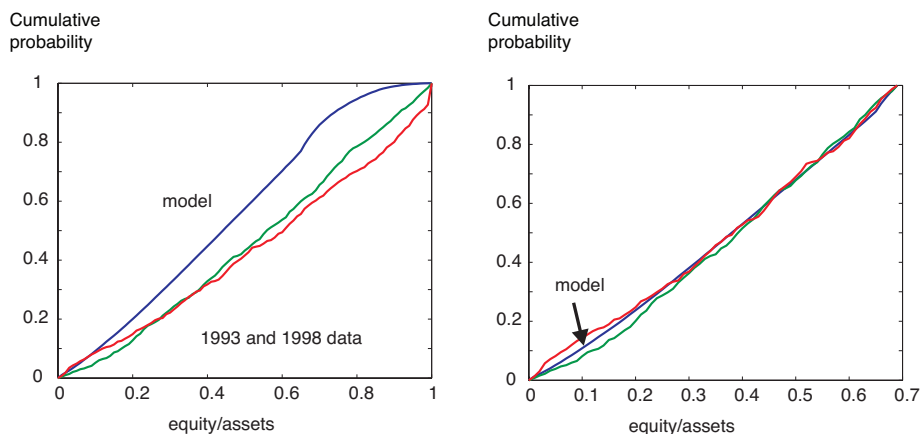


Figure 4: 1993 & 1998 SSBF data and model with tight constraint: capital structure cdfs

Table 5: Point estimates: Borrowing constrained model, unconstrained model & data

Parameter	Interpretation	Model with Endog. b	Model with Fixed b	Data
median $A\%$	median firm assets (size)	51.9	48.1	[43.1, 51.9]
consumption %	consumption as a fraction of net worth	3.2	3.6	3-5
default %	average default rate	6.4	4.4	3.5-4.5
neg. equity %	negative equity in the firm	16.1	10.6	15.7, 21.0

Table 5 shows that the model replicates successfully other targets. Median firm assets match well, as discussed above, and consumption is in the standard range.²⁸ The default point prediction for the model with the tight borrowing constraint is consistent with Fact 4 in section 2 that the average annual default rate is 3.5-4.5% on small business loans. The default prediction is higher for the model with the locally slack constraint. When the constraint is tight, 10.6% of firms have negative equity, which is below the empirical values for all firms of 15.7% in 1993 and 21.0% in 1998, see fact 5 in section 2. The value of 16.1% for the locally slack constraint is closer to these targets. In the next section we consider how these magnitudes vary with risk aversion.

²⁸Point estimates for the expected fraction of net-worth spent on consumption and the default probability are $c(\underline{\rho})\Phi_{\mu,\sigma}(\underline{\rho}) + \int_{\underline{\rho}}^{\infty} c(\rho)\phi_{\mu,\sigma}(\rho) d\rho$ and $\int_{\underline{x}}^{x^*(e,\rho)} f(x)\Phi_{\mu,\sigma}(\underline{\rho}) dx + \int_{\underline{\rho}}^{\infty} \int_{\underline{x}}^{x^*(e,\rho)} f(x)\phi_{\mu,\sigma}(\rho) dx d\rho$.

4.3.1 The Effect of Risk Aversion on Default and Negative Equity

In Section 3.3 we identified three effects that determine default: (i) The ex-ante level of debt \bar{v} ; (ii) the consumption loss from firm bankruptcy; and (iii) the wealth-to-firm-scale ratio. The first effect is the only one present in a static model, where a firm would default if and only if the return realization $x < \bar{v}$, i.e., if the firm has negative equity. The other two effects are a consequence of the value of continuing to operate a firm when equity is negative in a dynamic model.

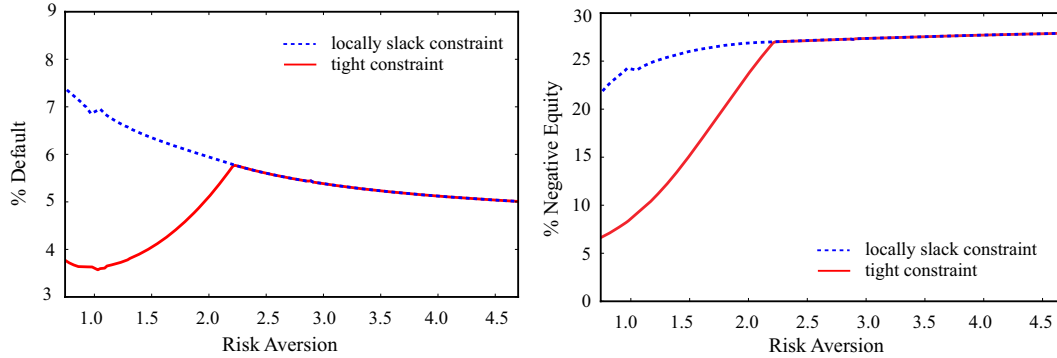


Figure 5: Model predicted default probabilities and negative equity as risk aversion varies

The left panel of figure 5 shows the default rate in our dynamic model.²⁹ In both panels the solid lines indicate the model with a tight credit constraint (where b is 21.5%) and the dotted lines correspond to the locally slack constraint. When an individual's risk aversion reaches about 2.2, the credit constraint is slack in both models and the dotted and solid lines coincide. The right panel shows the fraction of firms that have negative equity for a given level of risk aversion. This would correspond to the default rate in a static model, where dynamic effects (ii) and (iii) are inoperative. Our model indicates that these dynamic effects induce many firms with negative equity to continue to operate, thereby reducing the default rate. The gaps between the fraction of negative equity and the default rate are especially large for entrepreneurs with lower levels of ρ .

Figure 6 graphs the individual components of default. Recall from (10) that

$$x^* = \text{ex-ante debt} - \text{consumption loss} \times \text{wealth-to-firm-scale ratio}.$$

Theorem 3 proves the consumption hit from losing the firm is decreasing in risk aversion, which raises default cutoff x^* as ρ increases. We can see this monotonicity in the top left panel of figure 6. Theorem 4 proves the wealth-to-firm-scale ratio is increasing in risk aversion, which we also see in the figure. The product of these two terms is the dynamic effect (bottom right panel), which is

²⁹The graphs are not smooth near $\rho = 1$ due to roundoff errors as CRRA preferences converge to log utility.

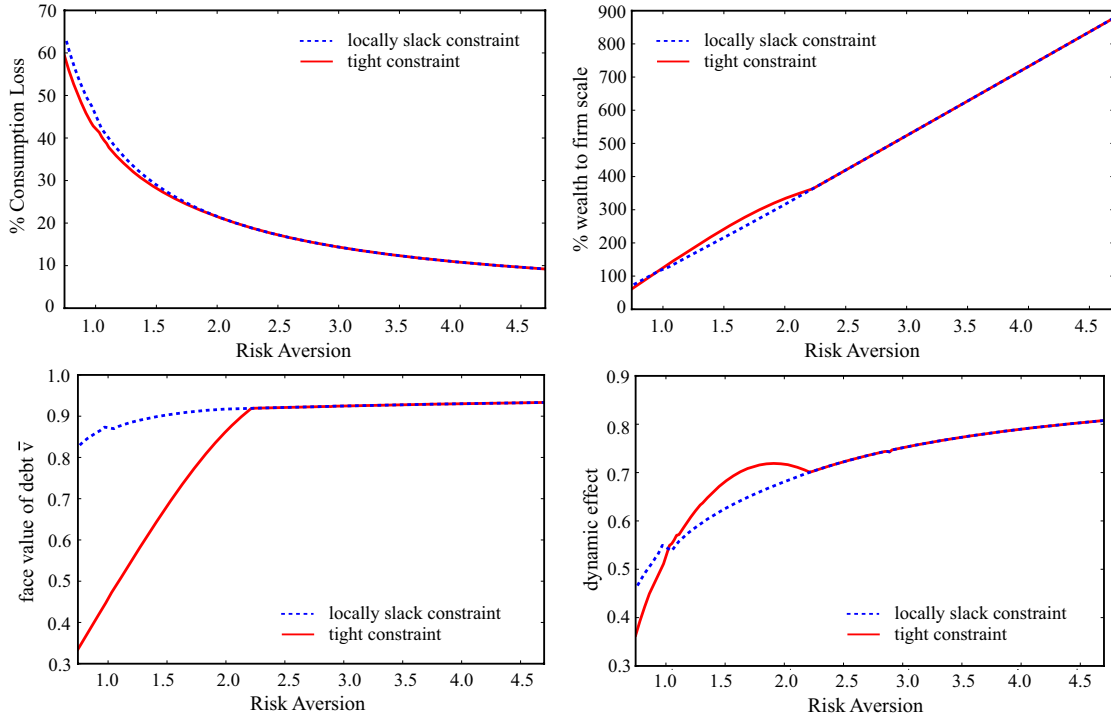


Figure 6: Default determinants: debt, consumption loss, wealth to size ratio & dynamic effect

theoretically ambiguous. For the calibrated model with the locally slack constraint the dynamic effect is increasing in ρ , i.e., it reduces default as ρ is increased. In the case of the tight constraint the dynamic effect is not monotone, but it is increasing for most ρ . The bottom left panel shows that \bar{v} increases with ρ , which results in an increase in negative equity and default. The lower default in the model with the tight constraint when ρ is low is primarily due to an entrepreneur's lower debt \bar{v} in this range. Compared to the slack constraint, the bottom right panel shows that the dynamic effect provides additional default reduction when risk aversion is between 1 and 2.2, and a slight increase for levels lower than 1, which results in the initial U-shape in the default probability in the left panel of figure 5.

4.3.2 The Effect of Risk Aversion on Loan Rates, Assets, Equity and Consumption

Theorem 4 proves that assets and the equity/asset ratio ϵ are decreasing in ρ , for sufficiently small ρ , for the model with the endogenous constraint. Figure 7 shows that these functions are indeed decreasing over the whole range of risk aversion for which we compute solutions. The intuition for the decrease in A and ϵ was discussed after the statement of Theorem 4: Both the reduction in project scale and the increased use of outside funds reduce the entrepreneur's exposure to risk.

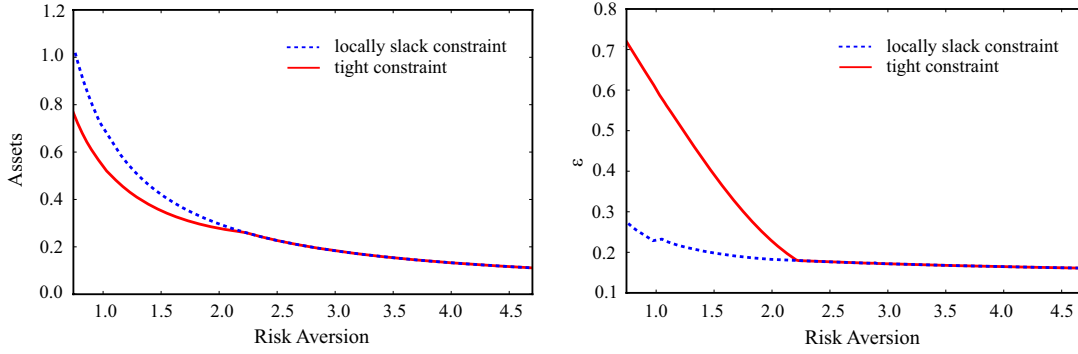


Figure 7: Effect of risk aversion on assets (A) and the equity/asset ratio (ϵ)

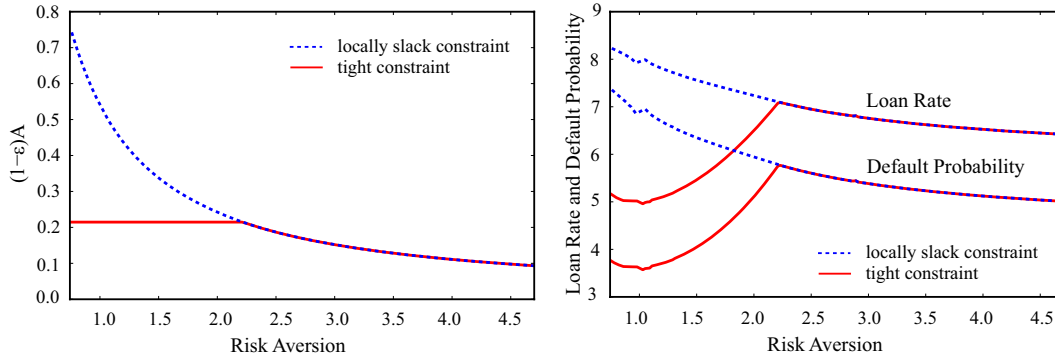


Figure 8: Effect of risk aversion on loans, $(1 - \epsilon)A$, and interest rates

Consider figure 8. The left panel shows the total amount of outside loans, $(1 - \epsilon)A$. Theorem 4 indicates the overall effect of risk aversion on loans is theoretically ambiguous since $1 - \epsilon$ increases, while A decreases. In our calibrated model, A decreases faster than $1 - \epsilon$ increases, and outside loans decrease.³⁰ When the constraint is tight, borrowing by entrepreneurs with sufficiently low risk aversion is constant and equal to $b = 0.215$. The right panel of figure 8 shows that loan rates track default rates. Note that the loan rate on a debt contract with face value \bar{v} and loan amount $1 - \epsilon$ is given by $r_L = \frac{\bar{v}}{1 - \epsilon} - 1$. In the data some firm returns are negative, which means we must adjust this rate. Formally, suppose some amount of debt κA in the initial investment is not observed, where $0 < \kappa < 1$ (see the bakery example in section 2). Then the total amount of debt is $(1 - \epsilon + \kappa)A$, resulting in an “adjusted” loan rate of $r_L^a = \frac{\bar{v}}{1 - \epsilon + \kappa} - 1$. Since the lender is risk-neutral, the present value of the negative returns is $\kappa = \frac{1}{1 + r_f} \int_{\underline{x}}^0 x dF(x)$.

³⁰Chen, Miao, and Wang (2010) study how risk aversion and idiosyncratic firm risk affect firm capital structure and default in a model with fixed scale and CARA preferences. The paper shows how the standard corporate finance approach breaks down when idiosyncratic risk cannot be diversified away. They find that borrowing rises with risk aversion, while it falls in figure 8.

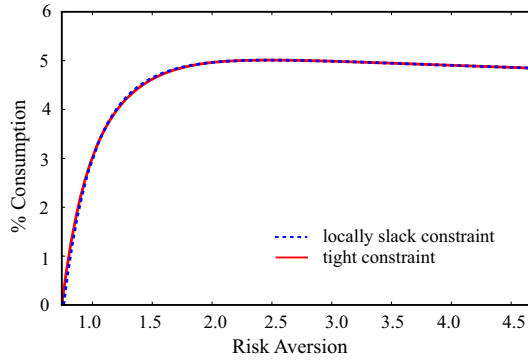


Figure 9: Effect of risk aversion on consumption

Figure 9 shows how consumption changes with risk aversion. The borrowing constraint has no effect on the outcome. At level of risk aversion $\underline{\rho}$, where solutions to the optimization problem start to exist, consumption is arbitrarily small. There are two reasons for this. First, the intertemporal elasticity of substitution is higher for lower ρ , which implies the entrepreneur is more willing to substitute current for future consumption. Second, the entrepreneur is more willing to bear risk and therefore forgoes current consumption to increase the project's scale.

4.4 Dynamics of Consumption, Net-worth and Assets

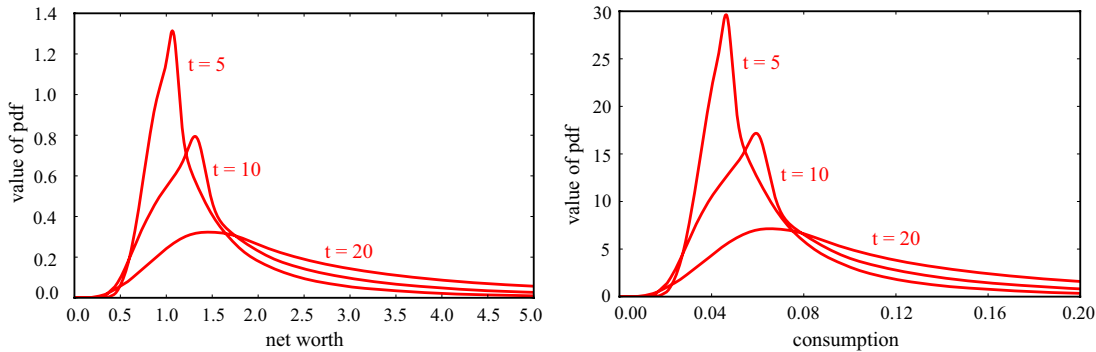


Figure 10: Pdf of net-worth and consumption for $\rho = 1.5$ after 5, 10, and 20 time periods

In this section we investigate the dynamics of entrepreneur consumption, net-worth and firm assets. In each time period, an entrepreneur receives a random project return, which determines consumption and net-worth in the following period. Figure 10 shows the density functions of the distribution of net worth and consumption after 5, 10, and 20 time periods for a person with risk

aversion $\rho = 1.5$, the model’s estimate for the median entrepreneur.³¹ The densities shown are for the case of the locally slack constraint — the predictions of the model with the tight constraint are very similar. The density functions for consumption and net-worth are closely related, since consumption is the same fraction of net-worth in all non-default states. At the start of the model, net worth is normalized to 1, but the density functions assign significant mass to points less than 1, which means that these unsuccessful entrepreneurs lost money. On the other hand, the densities have a “fat” upper tail, indicating that some entrepreneurs have the chance to be very successful.

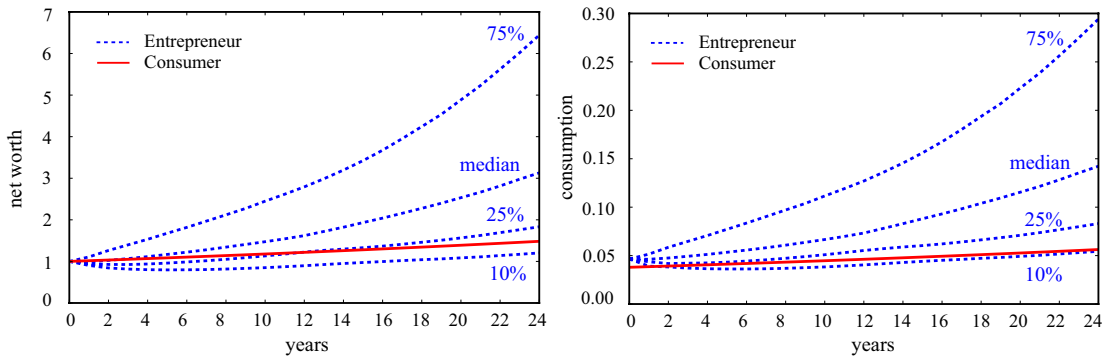


Figure 11: Entrepreneur & non-entrepreneur dynamics, $\rho = 1.5$: 10%, 25%, 50%, 75% quantiles

We next compare quantiles of the distributions of net-worth and consumption for two types of individuals with risk aversion $\rho = 1.5$: an entrepreneur with access to a production technology and a non-entrepreneur whose consumption and net worth are based entirely on a deterministic endowment.³² In figure 11, the solid lines show the deterministic net-worth and consumption profiles of the non-entrepreneur. The dashed lines are the 10%, 25%, median, and 75% quantiles of the distributions for entrepreneurs.³³ We find that the median entrepreneur has higher net-worth in all periods compared to the non-entrepreneur, and only the bottom 10% of entrepreneurs do worse. The wealth effect from having access to a project implies that all entrepreneurs have slightly higher consumption in the initial period, compared to non-entrepreneurs with the same utility function. Similarly, only the consumption of the bottom 10% of entrepreneurs is lower in subsequent periods, while the upside gains for the 75% quartile are substantial. The figure shows

³¹To determine the density functions we first use Monte Carlo simulations to generate firm returns. We then determine the density function by applying a Gaussian kernel density estimation to the generated data.

³²We assume the non-entrepreneur has access to the same loans as the entrepreneur to focus solely on the difference between being endowed or not with a production technology.

³³In each time period a different entrepreneur may be at a particular quantile. For example, a firm that ends up in the 75% after 20 years could have been at the bottom in the first few years and then experienced success in later years.

that entrepreneurs are very successful at minimizing the substantial downside project risks evident in the previous figure, while retaining access to the upside gains.

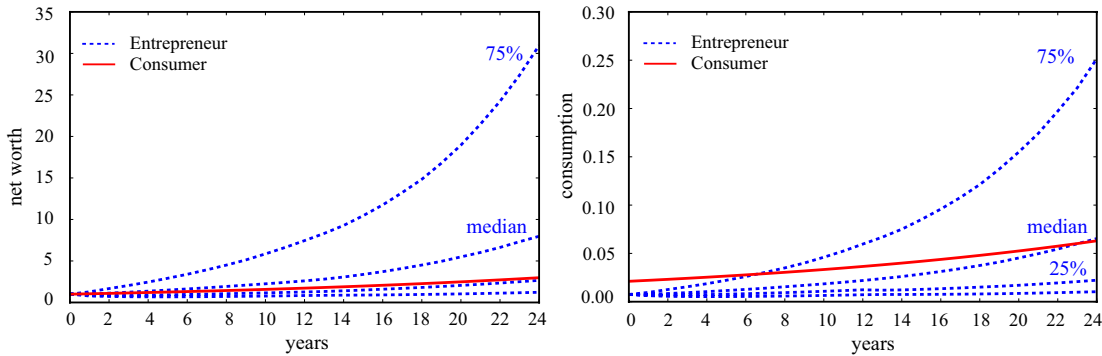


Figure 12: Entrepreneur & non-entrepreneur dynamics, $\rho = 0.8$: 10%, 25%, 50%, 75% quantiles

Figure 12 compares the dynamics of an entrepreneur and non-entrepreneur (solid line), when both have with low risk aversion of $\rho = 0.8$. Compared to figure 11, lower risk aversion results in entrepreneurs willing to run projects at a larger scale, thereby risking bigger losses. Now the net-worth of the bottom 25%, instead of just the bottom 10%, is below that of the non-entrepreneur for all 24 time periods. On the other hand, the most successful quartile of entrepreneurs has a net-worth in period 24 that is about 5 times higher than the $\rho = 1.5$ entrepreneurs. The lower risk aversion and increase in the intertemporal elasticity of substitution result in lower initial consumption for entrepreneurs. It takes 6 periods for the top quartile to catch up to the consumption of the non-entrepreneur, while the median entrepreneur must wait 24 periods. The median entrepreneur also ends up with net-worth that is only about two times that of the non-entrepreneur. If the entrepreneur had known at the outset that he would end up at the median, he likely would not have started the project. In contrast, for the top quartile entrepreneurship is a path toward wealth and high consumption. Thus, entrepreneurs with lower risk aversion are willing to significantly reduce their current consumption in the hope of a future reward that is far from certain. This attitude is reflected in comments made by David Siegel, CEO of Westgate Resorts, in a November 2012 E-mail to employees:³⁴ “I started this company over 42 years ago. At that time, I lived in a very modest home. We didn’t eat in fancy restaurants or take expensive vacations because every dollar I made went back into this company. Meanwhile, many of my friends... spent every dime they earned. They drove flashy cars and lived in expensive homes and wore fancy designer clothes. I

³⁴See <http://www.nbcnews.com/business/if-obama-re-elected-youll-be-fired-ceo-tells-workers-1C6385413?streamSlug=businessmain>

put my time, my money, and my life into this business—with a vision that eventually, some day, I too, will be able to afford to buy whatever I wanted.”

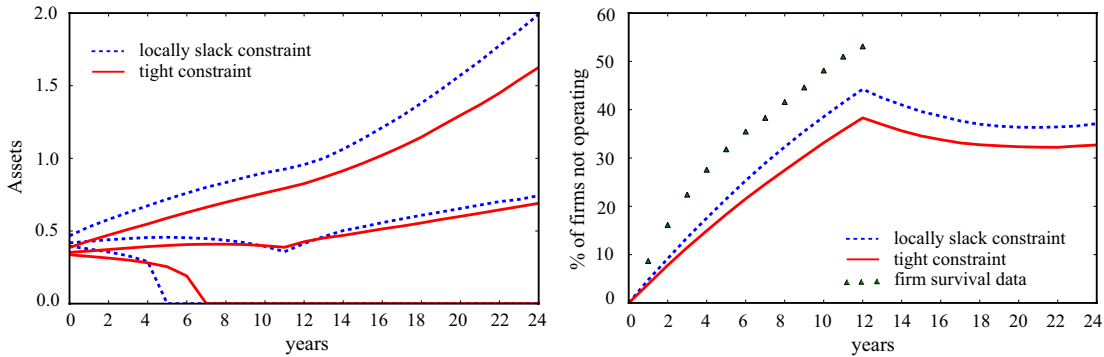


Figure 13: Dynamics of firm assets for $\rho = 1.5$.

Finally, the left panel of figure 13 shows the dynamics of assets for the median entrepreneur with $\rho = 1.5$ for the 25%, 50% and 75% quantiles. Firms in the 75% with a locally slack constraint are larger than those with a tight constraint. The median firm is about the same size under both regimes. For the bottom 25% firm size is zero after 5 periods with the locally slack constraint and after 7 periods for the tight constraint. Firm size becomes zero for 11 periods after default, when the firm is unable to operate. The fraction of firms that are not operating is shown in the right panel of figure 13. At $t = 0$ we start with a cohort of entrepreneurs with $\rho = 1.5$. The firms that default cannot operate at $t = 1$. At $t = 2$ firms that default are added to the number of those that do not operate. At $t = 13$ the cohort of firms that defaulted at $t = 1$ can again operate, which generates the peak of both curves at $t = 12$. Thus, up to $t = 12$ the graph shows the fraction of firms that do not survive, 44%, when the borrowing constraint is locally slack, and 38% when it is tight. At $t = 24$, about 37% of entrepreneurs are not operating a firm when the borrowing constraint is locally slack, and 33% with the tight constraint—the fractions in the long-run steady state are the same. In summary, in a world with slack credit constraints fewer potential entrepreneurs will operate firms, but those that do will operate at a larger scale.

According to BLS data,³⁵ the yearly survival rates of firms is remarkably stable over time. In their initial year, about 80% of firms survive. After being established for 5 years, the probability of surviving another year increases to about 91%, and to 95% after about 15 years. Thus, if one considers startups, about 30% of firms will be operating after 12 years. Since the SSBF

³⁵http://www.bls.gov/bdm/us_age_naics_00_table7.txt

data focuses on established firms rather than startups, we should expect the calibrated model's prediction to better fit firms that have already survived a few years. The triangles in figure 13 show the cumulative exit rates of firms that opened in 1994, conditional on surviving the first 5 years (e.g., the first data point is the fraction of such firms that exited between 2000 and 1999). The exit rates in the data are somewhat higher than those generated by the model. One reason for this difference may be that in practice some entrepreneurs quit for personal reasons rather than because their business fails, which is not captured by the model.

5 Concluding Remarks

Willingness to bear risk is often thought to be at the core of entrepreneurship. The goal of this paper is to understand how owner risk aversion affects small firms and how forward looking entrepreneurs use firm scale, capital structure, default and consumption to manage the significant business risks they face. We find that more risk averse entrepreneurs run smaller, more highly leveraged firms with more negative equity than their less risk averse counterparts. Less risk averse entrepreneurs are willing to forgo consumption to expand their firms in hope of future rewards that may never materialize. The empirical distributions we match (net worth, firm size, capital structure) show substantial heterogeneity, yet underlying differences in risk preference implied by the model are modest.

In static models of debt, firms default if and only if they have negative equity in the terminal period, and more leveraged firms therefore default at a higher rate. In our dynamic model firms have a continuation value, which in turn depends on the project's scale, the firm's financial structure and the owner's net-worth. To better understand default in a dynamic setting, we decompose the decision into its constituent parts: the level of indebtedness, the entrepreneur's ability to inject personal funds into the firm, and the entrepreneur's consumption loss from not being able to operate a firm, which influences the entrepreneur's willingness to avoid default. We analyze how the components vary with risk aversion and how the model matches data.

Our model is motivated by six facts. The first three facts collectively document that entrepreneurship is very risky. Fact 1 indicates that in the distribution of small firm returns, the real gross return on assets can be negative, but most returns are centered around the middle of the distribution and there is a long upper tail. Facts 2 and 3 indicate that owners tend to invest substantial personal funds in their firms and work at them, which compounds the risk. The model shows that firms use bankruptcy to avoid the extremely low returns that can occur, but consistent

with Fact 4, default is relatively low due to dynamic tradeoffs we analyze. Entrepreneurs use the firm's capital structure to avoid default and this is associated with the negative equity observed in SSBF data in Fact 5. Interestingly, Fact 6 documents that the distribution of firm capital structure in SSBF data is uniform. If firms were identical, we would expect to see Equity/Assets clustered around an optimal point. The uniform distribution observed in the data suggests that firms may face some underlying type of heterogeneity, and we focus on differences in willingness to bear risk. Overall our model indicates that firms use the choice of default, Equity/Assets, firm size and consumption to manage risk and these choices vary systematically with the level of risk.

We use a composite lender to aggregate the many sources from which firms obtain loans – banks, trade credit associations, leasing companies, and credit cards. In future work it would be useful to model the problems of these different lenders. For example, it would be instructive to consider the problem of a bank that must attract deposits and make loans, subject to default risk and regulation. Similarly, Eisfeldt and Rampini (2008) show that trade credit and leasing are important when lenders face information and enforcement problems, as is the case for small firms.³⁶ General equilibrium effects are also important in credit markets. Increased loan demand will raise the cost of external finance, which will offset some of the gains. We focus on idiosyncratic firm risk, which is particularly interesting in this setting, because firms are not tradable, and hence the owner cannot diversify this risk. Nonetheless, aggregate risk and correlated shocks would be interesting extensions to further explore the macroeconomic implications of the model.

³⁶Our lender can infer agents' risk aversion (e.g., from the loan request), thus adverse selection and moral hazard do not occur. Paulson and Townsend (2006) distinguish limited liability from moral hazard in a model of entrepreneurship, which would be another interesting extension in our model.

6 Appendix

6.1 Proof

Proof of Theorem 1. First, substitute $V_S(w) = w^{1-\rho}v_S$ and $V_B(w) = w^{1-\rho}v_B$ into the right-hand side of the objective of problem 1 and in constraint 2. We get

$$V_S(w) = \max_{C \geq 0, \mathcal{A} \geq 0, 0 \leq \epsilon \leq 1, \bar{v}} \left\{ u(C) + \beta \left[\int_{\mathfrak{B}} ((1+r)(w - \epsilon\mathcal{A} - C))^{1-\rho} v_B dF(x) \right. \right. \\ \left. \left. + \int_{\mathfrak{B}^c} (\mathcal{A}(x - \bar{v}) + (1+r)(w - \epsilon\mathcal{A} - C))^{1-\rho} v_S dF(x) \right] \right\}$$

Subject to: (1), (2), and (3)

Let $\lambda > 0$ and let current wealth be w . We must prove that $V_S(\lambda w) = \lambda^{1-\rho}V_S(w)$.

Suppose that the entrepreneur's wealth is λw and consumption is changed to λC , the firm's assets to $\lambda \mathcal{A}$, while ϵ remains unchanged. Then

$$\lambda^{1-\rho}v_B \left((1+r)(w - \epsilon\mathcal{A} - C) \right)^{1-\rho} = v_B \left((1+r)(\lambda w - \epsilon\lambda\mathcal{A} - \lambda C) \right)^{1-\rho}, \text{ and} \\ \lambda^{1-\rho}v_S \left(\mathcal{A}(x - \bar{v}) + (1+r)(w - \epsilon\mathcal{A} - C) \right)^{1-\rho} = v_S \left(\lambda\mathcal{A}(x - \bar{v}) + (1+r)(\lambda w - \epsilon\lambda\mathcal{A} - \lambda C) \right)^{1-\rho}.$$

This and (2) imply that bankruptcy set \mathfrak{B} remains unchanged. Thus, (1), (3) and the non-negativity constraints are satisfied. Next, note that the right-hand side of the objective changes by the factor $\lambda^{1-\rho}$. Because $V_S(\lambda w)$ is the maximum utility of the entrepreneur given wealth λw , it follows that

$$V_S(\lambda w) \geq \lambda^{1-\rho}V_S(w), \tag{18}$$

for all $\lambda > 0$. Thus, if we set $\tilde{w} = w/\lambda$ then

$$\lambda^{1-\rho}V_S(\tilde{w}) = \lambda^{1-\rho}V_S\left(\frac{1}{\lambda}\lambda\tilde{w}\right) = \lambda^{1-\rho}V_S\left(\frac{1}{\lambda}w\right) \geq \lambda^{1-\rho}\left(\frac{1}{\lambda}\right)^{1-\rho}V_S(w) = V_S(\lambda\tilde{w})$$

which, since \tilde{w} was chosen arbitrarily, implies that (18) holds with equality. Substituting $w = 1$ and $\lambda = w$ in (18) immediately implies that $V_S(w) = w^{1-\rho}v_S$. The proof that $V_B(w) = w^{1-\rho}v_B$ is similar.

Finally, it is straightforward to show that

$$V_S(w) = \frac{\ln(w)}{1-\beta} + v_S \\ V_B(w) = \frac{\ln(w)}{1-\beta} + v_B$$

■

Lemma 1 *Constraint 1 of Problem 1 binds.*

Proof of Lemma 1. Immediate: Suppose by way of contradiction that constraint (1) is slack. Then \bar{v} can be lowered thereby increasing the payoff in non-bankruptcy states, $V_S(\mathcal{A}(x - \bar{v}) + (1 + r)(w - \epsilon\mathcal{A} - C))$. Thus, the objective of problem 1 increases, a contradiction.³⁷ ■

Lemma 2 *Suppose that \mathfrak{B} is non-empty. Let*

$$x^* = \bar{v} - \left[1 - \left(\frac{v_B}{v_S} \right)^{\frac{1}{1-\rho}} \right] \frac{(1+r)(1-\epsilon A - c)}{A} \quad (19)$$

Then $\mathfrak{B} = \{x | \underline{x} \leq x < x^\}$. Conversely, if $x^* > \underline{x}$, then bankruptcy set \mathfrak{B} is non-empty.³⁸*

Note that when $\rho = 1$, i.e., log utility, we get $\exp(v_B)/\exp(v_S)$ instead of $(v_B/v_S)^{\frac{1}{1-\rho}}$ in (19).

Proof of Lemma 2. First consider the case where $\rho \neq 1$. If the entrepreneur chooses to default, the entrepreneur's continuation utility is

$$v^B = \left[(1+r)(1-\epsilon A - c) \right]^{1-\rho} v_B. \quad (20)$$

Otherwise, if the entrepreneur does not default, then continuation utility is

$$v^S(x) = \left[A(x - \bar{v}) + (1+r)(1-\epsilon A - c) \right]^{1-\rho} v_S. \quad (21)$$

Note that $x \in \mathfrak{B}$ if $v^B > v^S(x)$ and $x \notin \mathfrak{B}$ if $v^S(x) \geq v^B$.

Suppose that $v^S(x) \geq v^B$. We show that $v^S(x') > v^B$ for all $x' > x$. Note that

$$\frac{dv^S(x)}{dx} = \frac{(1-\rho)Av_S}{\left[(1+r)(1-\epsilon A - c) \right]^\rho} > 0,$$

since $v_S > 0$ for $\rho < 1$ and $v_S < 0$ for $\rho > 1$.

Thus, $v^S(x) - v^B \geq 0$ implies that $v^S(x') > v^B$ for all $x' > x$. Similarly, $v^B > v^S(x)$ implies $v^B > v^S(x')$ for all $x' < x$. Let x^* solve $v^B = v^S(x^*)$. Then the bankruptcy set is given by $\mathfrak{B} = \{x | \underline{x} \leq x < x^*\}$. (20) and (21) imply

$$\left[(1+r)(1-\epsilon A - c) \right] \left(\frac{v_B}{1-\rho} \right)^{1-\rho} = \left[A(x^* - \bar{v}) + (1+r)(1-\epsilon A - c) \right] \left(\frac{v_S}{1-\rho} \right)^{1-\rho},$$

³⁷The direct effect is to increase the entrepreneur's payoff by decreasing required payments to the lender and the indirect effect is to lower the bankruptcy probability.

³⁸At realization x^* , the entrepreneur is indifferent between default and continuing to operate the firm. Thus, (2) must hold with equality. Solving (2) for x^* implies (19).

which implies (19).

Now suppose that x^* is given by (19) and $x^* > \underline{x}$. Then by construction, $v^S(x^*) = v^B$. Further, the monotonicity result established above implies $v^B > v^S(x)$ for all $x < x^*$ and $v^S(x) \leq v^B$ for all $x \geq x^*$. Thus, the bankruptcy set is given by $\mathfrak{B} = \{x | \underline{x} \leq x < x^*\}$.

Finally, if $\rho = 1$ (i.e., log utility) then

$$v^S(x) = \frac{\ln(A(x - \bar{v}) + (1 + r)(1 - \epsilon A - c))}{1 - \beta} + v_S,$$

which is clearly increasing in x . Hence the same argument as above applies. ■

Proof of Theorem 2. Let $\Lambda(v_S)$ be the maximum entrepreneur utility in Problem 3. We must prove there exists v_S^* such that $\Lambda(v_S^*) = v_S^*$. First let $\rho > 1$. Suppose that $v_S = 0$. Then bankruptcy problem 2 and the fact that $u(c) < 0$ for utility normalization $u(c) = c^{1-\rho}/(1-\rho)$ implies $v_B < 0$. Further $u(c) < 0$. As a consequence, $\Lambda(0) < 0$. Now let \hat{v}_S be the entrepreneur's expected utility from autarky.

$$\hat{v}_S = \max_{c_0, c_1, \dots} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (22)$$

Subject to:

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \leq w \quad \text{and} \quad c_0, c_1, \dots \geq 0,$$

Note that if $v_S = \hat{v}_S$ and we choose $A = 0$ in problem 3, then we get the autarky utility \hat{v}_S . Thus, optimization implies that $\Lambda(\hat{v}_S) \geq \hat{v}_S$. Since Λ is continuous, the intermediate value theorem implies that there exists a fixed point v_S^* . Thus, $V_S(w) = v_S^* w^{1-\rho}$ solves the recursive problem.

For $\rho \leq 1$, re-normalize $u_\rho(x) = (x^{1-\rho} - 1)/(1 - \rho)$. Then $\lim_{\rho \rightarrow 1} u_\rho(x) = \ln(x)$.

Again it follows that $\Lambda(\hat{v}_S) \geq \hat{v}_S$. Recall that \bar{x} is the highest possible return. Clearly, expected utility is higher in an economy in which realization \bar{x} occurs with probability 1. In each period the entrepreneur can take a loan with interest rate r_f of size bw . Thus, the return on investment is given by $\bar{R} = \bar{x} + b(\bar{x} - (1 + r_f))$. The entrepreneur's problem is similar to the autarky problem above, except that we replace $1 + r$ by \bar{R} . If constraint $c_t \geq 0$ is slack, then the first order conditions immediately reveal that

$$c_t = (\beta \bar{R})^t c_0, \quad c_0 = 1 - \beta^{\frac{1}{\rho}} \bar{R}^{\frac{1-\rho}{\rho}}. \quad (23)$$

Clearly, there exists $\underline{\rho} < 1$ such that c_0 and hence $c_t > 0$ for $\rho \geq \underline{\rho}$.

Substituting (23) into the objective of (22) immediately implies that the resulting discounted utility $\bar{v}_S(\rho) < \infty$ for all $\rho > \underline{\rho}$. Since the entrepreneur's payoff under the actual project returns are lower, we get $\Lambda(\bar{v}_S(\rho)) \leq \bar{v}_S(\rho)$. Hence, the intermediate value theorem again implies that there is a fixed point v_S^* and hence $V_S(w) = v_S^* w^{1-\rho}$ solves the recursive problem.

■

Proof of Theorem 3. Let $C_{t,\rho}$, $t = 0, 1, \dots$, be the stochastic process that describes optimal consumption when risk aversion is ρ , starting with one unit of endowment at $t = 0$.³⁹ Let u_ρ be the CRRA utility function with relative risk aversion ρ .

Let c_ρ be given by

$$\sum_{t=0}^{\infty} \beta^t u_\rho(c_\rho) = \frac{u_\rho(c_\rho)}{1-\beta} = \sum_{t=0}^{\infty} \beta^t E_t[u_\rho(C_{t,\rho})] \quad (24)$$

Let $\rho < \sigma$. Then there exists a strictly increasing and strictly concave function h such that $h(u_\rho) = u_\sigma$. Thus, (24) implies

$$\begin{aligned} u_\sigma(c_\sigma) &= (1-\beta) \sum_{t=0}^{\infty} \beta^t E_t[u_\sigma(C_{t,\sigma})] = (1-\beta) \sum_{t=0}^{\infty} \beta^t E_t[h(u_\rho(C_{t,\sigma}))] \\ &< (1-\beta) \sum_{t=0}^{\infty} \beta^t h(E_t[u_\rho(C_{t,\sigma})]) < h\left((1-\beta) \sum_{t=0}^{\infty} \beta^t E_t[u_\rho(C_{t,\sigma})]\right) \\ &\leq h\left((1-\beta) \sum_{t=0}^{\infty} \beta^t E_t[u_\rho(C_{t,\rho})]\right) = h(u_\rho(c_\rho)) = u_\sigma(c_\rho). \end{aligned} \quad (25)$$

Thus, if we index v_S by ρ to show dependence of v_S on risk-aversion, (25) implies

$$c_\rho = \left(v_{S,\rho}(1-\beta)(1-\rho)\right)^{\frac{1}{1-\rho}}, \quad (26)$$

where c_ρ is strictly decreasing in ρ .

Next, consider the bankruptcy problem, which for $T = \infty$ reduces to

$$v_{B,\rho} = \max_{c_t} \sum_{t=0}^{\infty} \beta^t u_\rho(c_t), \quad \text{s.t.} \quad \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = 1 \quad (27)$$

The first order conditions of (27) immediately imply $\beta^t(1+r)^t u'_\rho(c_t) = \beta^{t+1}(1+r)^{t+1} u'_\rho(c_{t+1})$. Thus, $\beta(1+r) = 1$ implies that $c_t = c_{t+1}$, i.e., consumption is constant and independent of risk aversion σ , which we denote by c_B . Thus, (27) implies

$$c_B = \left(v_{B,\rho}(1-\beta)(1-\rho)\right)^{\frac{1}{1-\rho}}. \quad (28)$$

³⁹Upper case $C_{t,\rho}$ is the stochastic process and lower case c_ρ is the realization.

Thus, the consumption loss

$$1 - \left(\frac{v_{B,\rho}}{v_{S,\rho}} \right)^{\frac{1}{1-\rho}} = 1 - \frac{c_B}{c_{S,\rho}}, \quad (29)$$

is decreasing in ρ .

Note that the second statement of the theorem follows from continuity. ■

Proof of Theorem 4. Consider the case with the locally slack borrowing constraint. Thus, (1) is the only binding constraint, and we can use (1) to implicitly solve for \bar{v} as a function of A , ϵ , and c .

Let u_ρ be the CRRA utility function with risk aversion ρ . For any risk aversion parameters $\sigma > \rho$, there exists a strictly increasing and strictly concave function h such that $h(u_\rho) = u_\sigma$.

Using the definition of c_B and $c_{S,\rho}$ from the proof of Theorem 3, define

$$\theta(x; \rho, A, \epsilon, c) = \max \left\{ c_B \left[(1+r)(1-\epsilon A - c) \right], c_{S,\rho} \left[A(x - \bar{v}(A, \epsilon, c)) + (1+r)(1-\epsilon A - c) \right] \right\} \quad (30)$$

Then the optimal A , ϵ , and c must solve

$$\max_{A, \epsilon, c} \frac{u_\rho(c)}{\beta} + \int u_\rho(\theta(x; \rho, A, \epsilon, c)) dF(x) \quad (31)$$

We start with the first-order conditions with respect to A . To shorten notation, drop the arguments ϵ and c from θ . We get

$$\int u'_\rho(\theta(x; \rho, A_\rho)) \frac{\partial}{\partial A} \theta(x; \rho, A_\rho) dF(x) = 0, \quad (32)$$

where

$$\frac{\partial}{\partial A} \theta(x; \rho, A) = -(1+r)\epsilon c_B \mathbf{1}_{\{x < x^*(A)\}}(x) + \left[\left(x - \bar{v} - A \frac{\partial \bar{v}(A)}{\partial A} \right) - (1+r)\epsilon \right] c_{S,\rho} \mathbf{1}_{\{x \geq x^*(A)\}}(x). \quad (33)$$

Using the fact that $u_\sigma(\cdot) = h(u_\rho(\cdot))$, the optimal asset level A_σ satisfies

$$\int h'(u_\rho(\theta(x; \sigma, A_\sigma))) u'_\rho(\theta(x; \sigma, A_\sigma)) \frac{\partial}{\partial A} \theta(x; \rho, A_\sigma) dF(x) = 0. \quad (34)$$

Note that there exists \hat{x}_σ such that $\frac{\partial}{\partial A} \theta(x; \rho, A_\sigma) \leq 0$ for all $x \leq \hat{x}_\sigma$ and $\frac{\partial}{\partial A} \theta(x; \rho, A_\sigma) > 0$ for all $x > \hat{x}_\sigma$. Let $\bar{h} = h'(u_\rho(\theta(\hat{x}_\sigma; \sigma, A_\sigma)))$. Since h' is strictly decreasing, it follows that

$$\begin{aligned} 0 &= \int h'(u_\rho(\theta(x; \sigma, A_\sigma))) u'_\rho(\theta(x; \sigma, A_\sigma)) \frac{\partial}{\partial A} \theta(x; \rho, A_\sigma) dF(x) \\ &< \bar{h} \int u'_\rho(\theta(x; \sigma, A_\sigma)) \frac{\partial}{\partial A} \theta(x; \rho, A_\sigma) dF(x) \end{aligned} \quad (35)$$

We next show that decreasing σ to ρ raises the right-hand side of (35).

$$\begin{aligned}
& \frac{\partial}{\partial \sigma} \int u'_\rho(\theta(x; \sigma, A_\sigma)) \frac{\partial}{\partial A} \theta(x; \sigma, A_\sigma) dF(x) = \frac{\partial}{\partial \sigma} \int_{x^*}^{\bar{x}} u'_\rho(\theta(x; \sigma, A_\sigma)) \frac{\partial}{\partial A} \theta(x; \sigma, A_\sigma) dF(x) \\
& = \int_{x^*}^{\bar{x}} u''_\rho(\theta(x; \sigma, A_\sigma)) \frac{\theta(x; \sigma, A_\sigma)}{c_{S,\sigma}} \frac{\partial c_{S,\sigma}}{\partial \sigma} + u'_\rho(\theta(x; \sigma, A_\sigma)) \frac{\theta(x; \sigma, A_\sigma)}{\partial A} \frac{1}{c_{S,\sigma}} \frac{\partial c_{S,\sigma}}{\partial \sigma} dF(x) \\
& \quad - \frac{\partial x^*}{\partial \sigma} u'_\rho(\theta(x^*; \sigma, A_\sigma)) \left[\left(x^* - \bar{v} - A \frac{\partial \bar{v}(A)}{\partial A} \right) c_{S,\rho} - (1+r)\epsilon(c_{S,\rho} - c_B) \right] f(x^*) \\
& = \frac{\partial c_{S,\sigma}}{\partial \sigma} \int_{x^*}^{\bar{x}} u'_\rho(\theta(x; \sigma, A_\sigma)) \left(\frac{u''_\rho(\theta(x; \sigma, A_\sigma)) \theta(x; \sigma, A_\sigma)}{u'_\rho(\theta(x; \sigma, A_\sigma)) c_{S,\sigma}} + \frac{\theta(x; \sigma, A_\sigma)}{\partial A} \frac{1}{c_{S,\sigma}} \right) dF(x) \quad (36) \\
& \quad - \frac{\partial x^*}{\partial \sigma} u'_\rho(\theta(x^*; \sigma, A_\sigma)) \left[\left(x^* - \bar{v} - A \frac{\partial \bar{v}(A)}{\partial A} \right) c_{S,\rho} - (1+r)\epsilon(c_{S,\rho} - c_B) \right] f(x^*) \\
& = \frac{1}{c_{S,\sigma}} \frac{\partial c_{S,\sigma}}{\partial \sigma} \int_{x^*}^{\bar{x}} u'_\rho(\theta(x; \sigma, A_\sigma)) \left(-\rho + \frac{\partial}{\partial A} \theta(x; \sigma, A_\sigma) \right) dF(x) \\
& \quad - \frac{\partial x^*}{\partial \sigma} u'_\rho(\theta(x^*; \sigma, A_\sigma)) \left[\left(x^* - \bar{v} - A \frac{\partial \bar{v}(A)}{\partial A} \right) c_{S,\rho} - (1+r)\epsilon(c_{S,\rho} - c_B) \right] f(x^*)
\end{aligned}$$

Theorem 3 proves that increasing σ lowers $c_{S,\sigma}$. Hence $\frac{\partial c_{S,\sigma}}{\partial \sigma} < 0$. Further, (35) and $\frac{\partial}{\partial A} \theta(x; \sigma, A_\sigma) < 0$ for $x < x^*$ implies

$$0 < \int_0^{x^*} u'_\rho(\theta(x; \sigma, A_\sigma)) \frac{\partial}{\partial A} \theta(x; \sigma, A_\sigma) dF(x) < \int_{x^*}^{\bar{x}} u'_\rho(\theta(x; \sigma, A_\sigma)) \frac{\partial}{\partial A} \theta(x; \sigma, A_\sigma) dF(x).$$

Thus,

$$\frac{1}{c_{S,\sigma}} \frac{\partial c_{S,\sigma}}{\partial \sigma} \int_{x^*}^{\bar{x}} u'_\rho(\theta(x; \sigma, A_\sigma)) \left(-\rho + \frac{\partial}{\partial A} \theta(x; \sigma, A_\sigma) \right) dF(x) < 0,$$

for all sufficiently small ρ . Further, if the default probability is sufficiently small, then the single-peakedness of the return distribution ensures that $f(x^*)$ is small. Thus, the derivative in (36) is negative. This, and (35) imply

$$\int_{x^*}^{\bar{x}} u'_\rho(\theta(x; \rho, A_\sigma)) \frac{\partial}{\partial A} \theta(x; \rho, A_\sigma) dF(x) > 0. \quad (37)$$

At the solution of the optimization problem, the objective must be locally concave. At A_σ the partial derivative is positive, i.e., in order to obtain the optimum we must increase A from A_σ , hence $A_\rho > A_\sigma$, i.e., assets are decreasing in risk aversion.

The proof that ϵ decreases with risk aversion is similar. To simplify notation we drop c and A in function θ to get,

$$\frac{\partial}{\partial \epsilon} \theta(x; \rho, \epsilon) = -(1+r)A c_B \mathbf{1}_{\{x < x^*(\epsilon)\}}(x) + \left[A \frac{\partial \bar{v}(A)}{\partial \epsilon} - (1+r)A \right] c_{S,\rho} \mathbf{1}_{\{x \geq x^*(\epsilon)\}}(x). \quad (38)$$

The first order condition for an entrepreneur with risk aversion ρ is

$$\int u'_\rho(\theta(x; \rho, \epsilon_\rho)) \frac{\partial}{\partial \epsilon} \theta(x; \rho, \epsilon_\rho) dF(x) = 0, \quad (39)$$

and for risk aversion σ is

$$\int h'(u_\rho(\theta(x; \sigma, \epsilon_\sigma))) u'_\rho(\theta(x; \sigma, \epsilon_\sigma)) \frac{\partial}{\partial \epsilon} \theta(x; \rho, \epsilon_\sigma) dF(x) = 0. \quad (40)$$

Note that (33) implies that $\frac{\partial}{\partial \epsilon} \theta(x; \rho, \epsilon) < 0$ for all default states. Thus, for (40) to be satisfied, $\frac{\partial}{\partial \epsilon} \theta(x; \rho, \epsilon) > 0$ for all non-default states.

The same argument as in (35) can therefore be applied to prove that

$$\int u'_\rho(\theta(x; \sigma, \epsilon_\sigma)) \frac{\partial}{\partial \epsilon} \theta(x; \sigma, \epsilon_\sigma) dF(x) > 0 \quad (41)$$

Thus, we must increase ϵ from ϵ_σ to get to optimal value ϵ_ρ . Hence ϵ is decreasing in risk aversion.

■

6.2 Construction of the Distribution of Firm Returns

Herranz, Krasa, and Villamil (2009) use the 1993 SSBF to compute the ROA because it includes interest payments. They exclude unincorporated firms because SSBF data do not account for the entrepreneur's wage from running the firm. They assume that all firms have access to the same constant returns to scale blue print technology, thus the return per unit of asset for a particular firm is a sample point from the distribution of this technology. The firm's nominal after-tax ROA is:

$$x = \frac{\text{Profit after taxes} + \text{Interest Paid}}{\text{Assets}} + 1. \quad (42)$$

Interest paid is added to after tax profit because the ROA must include payments to debt and equity holders.⁴⁰ The nominal rate is adjusted by 3% for inflation (BLS CPI 1993). ROA is computed instead of ROE because many firms had negative equity (about 16% in the 1993 SSBF and 21% in 1998). Many of these firms stay in business because owners use personal funds to "bail out the firm." Computing a ROA and modeling owners' allocations of equity and debt accounts for this.⁴¹

⁴⁰We use after tax returns as this is relevant for an entrepreneur to decide how much net-equity to invest.

⁴¹Computing ROE is misleading for firms near distress. For firms with low but positive equity, small profit gives a high percentage return. Also, many loans are collateralized; book value of equity understates owner contribution (the "correct" value of equity).

6.3 Numerical Procedure

Given model parameters, compute solutions to problem 3 as follows. For fixed v_S , use the first order conditions to solve for the optimum. (5) is always slack, since $c + \epsilon A = 1$ would imply zero future consumption. We need only verify if (6) and (or) the non negativity constraints bind by checking for positive Lagrange multipliers in the first order conditions. Inserting the solution of the first order conditions into the objective yields $\Lambda(v_S)$. To find a fixed point, compute slope $\Lambda'(v_S)$ by the Envelope Theorem or compute the difference of Λ between v_S and a point v'_S , giving solution ϵ, A, c, \bar{v} . Section 4.1 explains how to go from these point estimates to cdfs. Compute $\underline{\rho}$ from the first order condition using the fact that $v_S \rightarrow \infty$ as $\rho \downarrow \underline{\rho}$.⁴²

Table 6: Comparative statics for T : Fix $r_f = 1.2\%$, $r = 4.5\%$, $\beta = 0.97$, $\delta = 0.10$

T	6	7	8	9	10	11	12	13	14	15	16	20
med $A\%$	56.2	54.4	52.7	51.1	49.6	48.1	46.7	45.4	44.3	43.3	42.4	38.9
default%	6.1	5.6	5.3	5.0	4.7	4.4	4.2	4.0	3.8	3.6	3.5	2.9
cons.%	3.7	3.7	3.7	3.7	3.7	3.6	3.6	3.6	3.6	3.6	3.6	3.6
neg eq%	8.4	8.8	9.3	9.7	10.0	10.6	11.1	11.9	13.4	15.1	17.0	21.0

6.4 The Impact of Bankruptcy Policy: T, δ

Bankruptcy Exclusion Period T : Consider the effect of changes in T on the owner's payoff, where longer exclusion raises the penalty of bankruptcy. Table 6 fixes μ, σ, b , and evaluates the effect of altering the exclusion period from the benchmark $T = 11$ for the model with the tight constraint. As T decreases default increases rapidly. Firm size increases, measured by median asset level A . Because b is fixed, the decrease in total investment results in a decrease in equity and an increase in debt, which raises negative equity. One of the main economic arguments in support of recent U.S. bankruptcy reform was that more stringent bankruptcy rules lower interest rates, and therefore help borrowers. The loan rate indeed decreases as T increases. However, stricter bankruptcy provides less insurance against bad realizations, and this effect dominates. Table 7 shows that lowering the exclusion period increases the owner's payoff, and the model implies that it is optimal to set T as low as possible. Decreasing T is beneficial because it allows a firm to restart and be productive, in accordance with the historical rationale for bankruptcy, though $T = 0$ may

⁴²Choose a large value for v_S , solve for the remaining parameters including ρ , which approximates $\underline{\rho}$. In other words, rather than solving the fixed point problem for v_S , solve it for ρ .

Table 7: Effects as T Varies: % change in net-worth compared to benchmark

risk aversion ρ	0.9	1.2	1.5	1.8	2.1	2.5	3.0	3.5	4.0
$T = 6$	36.9	11.2	7.7	6.1	5.0	3.9	3.1	2.6	2.2
$T = 8$	19.8	5.5	3.9	3.0	2.4	1.9	1.5	1.3	1.1
$T = 10$	6.3	1.3	1.1	0.8	0.7	0.5	0.4	0.3	0.3
$T = 11$	—	—	—	—	—	—	—	—	—
$T = 12$	-3.6	-0.7	-0.9	-0.7	-0.5	-0.4	-0.3	-0.3	-0.2
$T = 14$	-7.6	-4.4	-2.1	-1.8	-1.4	-1.0	-0.8	-0.6	-0.5
$T = 16$	-12.7	-6.5	-3.8	-2.4	-2.0	-1.5	-1.1	-0.9	-0.7

not be possible or desirable.⁴³ The downside of lower a T is increased risk absorbed by lenders such as banks, whose own risk of default increases, a cost not accounted for in our model.

The tradeoff between insurance provided by firm bankruptcy and higher interest rates induced by increased default has been analyzed for consumer bankruptcy by Chatterjee, Corbae, Nakajima, and Rios-Rull (2007) and Livshits, MacGee, and Tertilt (2007). In both models consumers trade off insurance against health, divorce or family shocks versus consumption smoothing; the signs of the tradeoffs differ but the welfare effects are modest.⁴⁴ Meh and Terajima (2008) add unincorporated entrepreneurs to the model, and find a larger welfare effect of 1.78%. In contrast, table 7 reports strong effects from reducing the exclusion penalty in our model, particularly for agents with low levels of risk aversion. The main reason for the difference between our model of firm bankruptcy and the consumer bankruptcy models is that reducing the punishment period encourages entrepreneurs to invest more in their firms and operate at a larger scale, which increases output. In this sense, even though we do not find extreme variations in ρ , risk interacts with the dynamic decision problem, return distribution and bankruptcy rules to have an important effect on some (heterogeneous) agents, namely those that invest most heavily in their firms.

Bankruptcy Cost δ : Changing the costs of bankruptcy has only a minor effect on the model's

⁴³We have complete information, but information frictions would make a very low T undesirable. Suppose entrepreneurs could choose between the blueprint return distribution and an alternative with more risk than is socially desirable. In an institutional environment in which strong ex ante and interim screening mechanisms exist and penalties are credible, a small T can be sufficient to avoid moral hazard or adverse selection. In contrast, a country with poor institutions would require a larger T to deter entrepreneurs from choosing the alternative distribution, thus generating additional inefficiencies.

⁴⁴In our model credit is secured, for example by a house, and "bad luck" is a poor return x rather than the health, job, divorce or family shocks in the consumer models. The first paper finds that when punishment is reduced from 10 to 5 years welfare drops by 0.05%, thus the negative effect from a higher interest rate and tighter borrowing constraint slightly dominates the insurance benefit of a shorter punishment period. The second paper shows that the insurance effect is sometimes weakly dominant, but again the effect is modest.

endogenous parameters. However, if δ is very large and there are large fixed costs to creditors to recover payments in default, agents will try to avoid costly bankruptcy, through debt forgiveness or renegotiation. The static model of Krasa, Sharma, and Villamil (2008) shows that when courts are sufficiently inefficient substantial deadweight losses are possible.

6.5 Counterfactual Exercise: Empirical vs. Normal Returns

The features of return distribution $f(x)$ are important for understanding entrepreneur behavior. We conduct two counterfactual experiments to show that the return distribution is important. The experiments replace the empirical ROA distribution computed from SSBF data, keeping all other benchmark settings the same, with two different normal distributions.

Best Fit Normal Distribution. Let $\phi_{\mu,\sigma}$ be the density of a normal distribution with mean μ and standard deviation σ and f be the density of the SSBF distribution. Solve $\min_{\mu,\sigma} \sup_x |\phi_{\mu,\sigma}(x) - f(x)|$ to find a normal distribution that best approximates the empirical density function. The resulting values are $\mu = 1.193$ and $\sigma = 0.394$. In order to fit the “middle” this normal distribution has less mass in the tails and, as a consequence, is less risky. Thus, when re-calibrating the model, median risk aversion increases from 1.55 to 2.33 but at the same time, for given ρ , the lower project risk in this normal distribution encourages entrepreneurs to run larger firms. Default is lower, again because this normal distribution has a thinner lower tail. Finally, the thinner upper tail implies that less firms will be “lucky” and have a very good realization. In order to match the distribution of net-worth invested, firms must be more leveraged: Given two solvent firms with the same realization, a more leveraged firm earns a higher return because the owner receives a higher residual after making the fixed debt payment.⁴⁵ The somewhat higher level of debt also implies that more low realizations will result in negative equity, and the predicted percentage of firms with negative equity increases from 10.6% to 13.7%.

Normal Distribution with SSBF μ, σ . Table 8 compares the empirical SSBF pdf with a normal distribution with the same mean and standard deviation. The table shows the results for this normal distribution are significantly at odds with the data, highlighting the importance of the return distribution. First, the fat tails lead to μ and σ with all point mass at $\underline{\rho}$ and $\bar{\rho}$, where $\bar{\rho}$ is the highest risk aversion for which we compute a solution. Generally, we can choose $\bar{\rho}$ sufficiently high that the mass above $\bar{\rho}$ is negligible; this cannot be done for this normal distribution with fat tails and $\bar{\rho}$ affects the results.⁴⁶ Second, the model predictions in the last column of table 8 are implausible.

⁴⁵This also explains the higher value of b .

⁴⁶Upper bound $\bar{\rho}$ is needed for computation; it is impossible to compute solutions for a fine grid $[\underline{\rho}, \infty]$.

Table 8: Counterfactual Experiment: Normal Distributions

Parameter	Data	Empirical $f(x)$ SSBF 1993	Best Fit Normal $\mu=1.193, \sigma=0.3938$	Normal $\mu=1.300, \sigma=1.193$
μ	1-3	1.55	2.33	$4.4 * 10^8$
σ	NA	.83	1.11	$7.9 * 10^8$
b%	NA	21.5	30.0	23.4
fit	NA	0.042	0.040	.045
median A%	[43.1,51.9]	48.1	54.7	38.6
default %	3.5	4.4	1.5	61.0
cons. %	3-5	3.6	4.9	3.1
neg. Eq %	15.7	10.6	13.7	64.4

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