# Debt Contracts and Cooperative Improvements

Stefan Krasa* Tridib Sharma† Anne P. Villamil ‡

February 9, 2004

**Abstract**

In this paper we consider a dynamic game with imperfect information between a borrower and lender who must write a contract to produce a consumption good. In order to analyze the game, we introduce the concept of a coalitional perfect Bayesian Nash equilibrium (cPBNE). We prove that equilibria exist and are efficient in a precise sense. Deterministic contracts that resemble debt are optimal for a general class of economies. The cPBNE solution concept captures both the non-cooperative aspect of firm liquidation and the cooperative aspect of firm restructuring.

**JEL Classification Numbers:** C70, D60, G30, K40

*Keywords:* Debt Contract; Incomplete Information; Bankruptcy; Renegotiation; Liquidation; Efficiency; Fairness; Enforcement; Cooperative; Non-cooperative

---

*Department of Economics, University of Illinois, 1206 South 6th Street, Champaign, IL 61820 USA, http://www.staff.uiuc.edu/˜skrasa

†Centro de Investigación Económica, ITAM, Ave. Camino Santa Teresa #930, México, D.F. 10700, sharma@itam.mx

‡Department of Economics, University of Illinois, 1206 South 6th Street, Champaign, IL 61820 USA, avillami@uiuc.edu

We thank Ludovic Renou, Nicholas Yannelis and an anonymous referee for useful comments. Krasa and Villamil gratefully acknowledge financial support from NSF grant SES-031839 and the Center for Private Equity Research at the University of Illinois.

# 1 Introduction

There are two main ways for troubled firms to resolve financial distress: liquidation and renegotiation.[1] When a distressed firm is liquidated, it is shut down and any remaining assets are distributed to creditors. When renegotiation occurs, agents use information acquired after the contract was signed to modify the initial agreement if it is mutually beneficial to do so. We consider a multi-stage game of imperfect information where both liquidation and renegotiation are possible and analyze two questions: What types of financial contracts will agents write when both ways of resolving distress are possible? In what sense are these contracts optimal (i.e., efficient)? There are large literatures on bankruptcy, renegotiation and concepts of efficiency when information is imperfect. We begin by making clear the precise relationship between this paper and recent work on liquidation and renegotiation. We defer to Section 6 a broader discussion of efficiency when agents' actions reveal information.

The paper builds on Krasa and Villamil [12], which proposes a costly enforcement game where a firm and an investor write a contract to facilitate production.[2] The firm obtains funds from the investor in exchange for a repayment promise. After investment, the firm privately observes the return and chooses whether or not to repay. If the firm does not repay, agents have the opportunity to renegotiate the contract. Otherwise, the investor can seek costly enforcement of the promised payment (and liquidation of the firm). We generalize Krasa and Villamil [12] along two dimensions.

1. Because agents are rational, the possibility of renegotiation is foreseen ex-ante. Enforcement will therefore occur only if there are no benefits from renegotiation, i.e., the contract is renegotiation proof. Krasa and Villamil specify a sufficient condition for contracts to be renegotiation proof, but do not show how the constraint is derived from a game. We introduce a solution concept, coalitional perfect Bayesian Nash equilibrium (cPBNE), which derives this requirement as a necessary and sufficient condition for renegotiation proofness.

2. Krasa and Villamil prove that the entrepreneur's default and investor's enforcement decisions are both deterministic, i.e., the firm's repayment decision is not random given the observed return and the investor's enforcement decision depends solely on the amount repaid. Examples of deterministic contracts include debt (or simple

[1] There are several different notions of bankruptcy. Liquidation is a formal legal process (e.g., Chapter 7 of the U. S. Bankruptcy Code) and renegotiation may occur either as a court supervised "workout" or a private agreement. The key idea in renegotiation is that agents will exploit any mutually beneficial improvements that are common knowledge.

[2] There are two key differences between the enforcement model and standard contracting models (e.g., Gale and Hellwig [4] or Williamson [16]): costly enforcement is a choice variable and there is limited commitment to the ex ante contract.

debt), which are contracts where the debtor announces default if and only if the required payment cannot be met and the creditor requests enforcement if and only if such a default occurs. Krasa and Villamil further prove that if simple debt is renegotiation proof, it is the optimal contract because it minimizes the incidence of default.[3] Sharma [15] considers the case where simple debt contracts need not be renegotiation proof. He constructs an example in which simple debt is not optimal because a tradeoff exists between minimizing default and weakening the renegotiation proofness constraint. This paper provides a complete analysis of the structure of optimal contracts in this setting.

In order to provide a game-theoretic foundation for Krasa and Villamil's [12] renegotiation proofness condition, we introduce the concept of a coalitional perfect Bayesian Nash equilibrium (cPBNE). Formally, a cPBNE of our two-player game is a PBNE in which both agents cannot benefit from deviating at any information set. There are three time periods, $t = 0, 1, 2$, in which agents could jointly deviate. Absence of such deviations at $t = 0$ and $t = 1$ corresponds to the requirement that the decision rules are ex-ante and interim efficient, respectively. At $t = 2$, we show that it is not possible for full information revelation to occur because this would destroy the investor's incentive to enforce. This, in turn, would give the entrepreneur the incentive to lie and make the lowest possible payment. Thus the absence of improvements at $t = 2$ does not correspond to ex-post efficiency, but rather to a second interim period in which the investor has updated his prior using information that was endogenously and optimally revealed by the entrepreneur's payment action at $t = 1$. Forges [3] refers to this as posterior efficiency. Intuitively, posterior efficiency means that agents have no incentive to renegotiate the contract given the information that was revealed earlier in the game. We provide an extended discussion of efficiency concepts under incomplete information in Section 6.

Our coalitional solution concept has both cooperative and non-cooperative aspects. Von Neumann and Morgenstern's classic book considered both approaches to games, and commitment was central to the dichotomy. Games in which agents cannot make commitments to coordinate their strategies are noncooperative games. Analogously, games in which players can commit to coordinate strategies are cooperative. The strategic or non-cooperative approach requires a complete description of the rules of the game so that the strategies available to the players can be analyzed, and the goal is to find equilibrium strategies for each player. In contrast, cooperative games apply to situations where players can negotiate before the game is played and that commitments made in these negotiations are binding. In this case the strategies are not the main object of interest, but rather the determinants of the set of feasible contracts. Because limited commitment is central to financial contract-

[3] Simple debt corresponds to a standard loan contract. For non-bankruptcy realizations the borrower makes a fixed repayment (principal plus interest) that is not contingent on the realized state of nature. When bankruptcy occurs, the creditor receives all assets that can be legally transferred.

ing problems (i.e., we take into account agents' opportunities to deviate at later stages of the game from strategies specified at the outset), elements from both cooperative and non-cooperative game theory exist. For example, information revelation by the entrepreneur is non-cooperative, but the selection of the enforceable contract is cooperative and efficient in a sense we will make precise. See the discussion in Section 2.4.

There has been increasing interest in applying cooperative game theory to models with differential information.[4] With the exception of Ichiishi and Sertel [8], the papers in this literature consider static exchange economies where information is exogenously revealed to agents in an interim period but agents cannot revise their choices. In contrast, we consider a dynamic model where at $t = 2$ the investor uses the information revealed by the entrepreneur's payment decision at $t = 1$ to update his belief. Thus, information revelation is endogenous (but not complete) in our model. Our endogenous information revelation differs from learning in differential information economies in two main respects (see, for example, Koutsougeras and Yannelis [10]). First, in these learning models information is exogenous (i.e., it is not revealed optimally by agents' actions as part of a PBNE as in our model). Second, private information may be fully revealed in the limit. In contrast, it is not possible to fully reveal all private information in our model because this would destroy the investor's incentive to enforce, thus unravelling the equilibrium as explained above.

In order to make clear the relationship between our model and the large literature on economies with differential information, we call $t = 0$ the (standard) ex ante period, but augment the notion of the interim period. We call period $t = 1$ the exogenous interim period because information is privately revealed to the entrepreneur (i.e., the state). After the entrepreneur privately observes the state, he has the opportunity to make a payment to the investor by placing "money on the table," if it is optimal to do so. This payment reveals information about the state to the investor, which allows the investor to update his belief in period $t = 2$. We call this the endogenous interim period because the investor uses the information that was optimally revealed (as part of a PBNE) by the entrepreneur's payment. Given the updated belief, the investor then decides whether it is optimal to retain only the money on the table (if any) or to proceed with costly enforcement.

# 2 The Model

## 2.1 Timing of Decisions

Consider a three period economy with a risk-neutral investor and entrepreneur, and a court that can enforce contracts if requested to do so. The investor is endowed with the input that

[4] See Yannelis [17] for an early contribution. Also, Koutsougeras and Yannelis [9] analyze the core and Krasa and Yannelis [13], [14] analyze the Shapley value in economies with differential information. They propose the respective solution concepts as alternatives to the rational expectations equilibrium.

is essential for production but no technology, and the entrepreneur owns the production technology but has no input. One unit of input is required to produce the output, which is described by a random variable with finitely many realizations $x \in X = \{\underline{x}, \ldots, \bar{x}\} \subset \mathbb{R}_+$. Ex-ante the agents have a common prior $\beta(.)$ over $X$ with $\beta(x) > 0$, and the investor and entrepreneur value only this final period output.[5] The timing of events is as follows:

**t=0 (ex ante):** To produce, the entrepreneur and investor must trade. This is done by specifying an enforceable loan contract $\ell(x, v) \geq 0$, which is a payment schedule with state $x$ determined by the court at $t = 2$ and a payment $v \geq 0$ made by the entrepreneur at $t = 1$. If the entrepreneur and investor cannot agree on a payment schedule, no investment occurs and each agent receives a zero reservation utility.

**t=1 (exogenous interim):** The entrepreneur, but not the investor, privately observes output realization $x$ and selects a payment $v \geq 0$ that cannot be retracted. Payment $v$ is not enforceable by the court (though the enforceable payment $\ell(\cdot)$ depends on $v$). Because it is not enforceable, we refer to $v$ as a voluntary payment.

**t=2 (endogenous interim):** The investor chooses whether to request costly enforcement by the court. If no enforcement is requested, the investor's payoff is $v$ and the entrepreneur's payoff is $x - v$. If enforcement is requested, the investor pays cost $c$, the court determines the true state $x$, and the court enforces payment $\ell(x, v)$ if it is in the cone of "fair payments" specified in definition 2 below. With enforcement, the investor and entrepreneur respective payoffs are $v + \ell(x, v) - c$ and $x - v - \ell(x, v)$.

## 2.2 Definition of the Perfect Bayesian Nash Equilibrium

Given contract $\ell(x, v)$, the entrepreneur's payment decision (and the information it reveals) and the investor's enforcement probability $e_v$ must fulfill equilibrium restrictions. The entrepreneur's payment $v$ must be chosen optimally, the investor's enforcement decision must be chosen optimally, and belief $\beta'_v$ must be derived by Bayes' rule. We formalize this idea in definition 1 below. We allow the investor and the entrepreneur to randomize over actions, if it is optimal to do so. The entrepreneur's choice of payment $v$ is described by strategy $\mathcal{v}_x$. For simplicity of exposition we assume that these payments $v$ are restricted to a countable set $V$,[6] where $\mathcal{v}_x(v)$ is the probability that payment $v$ is chosen, given that the entrepreneur observes realization $x$. The investor knows $\mathcal{v}_x$, the rule that the entrepreneur uses to choose payment $v$, and uses it to update his prior from $\beta$ to $\beta'_v$. Finally, the investor selects a probability of enforcement, $e_v$, given payment $v$ from the entrepreneur. Thus, $\mathcal{v}_x$ and $e_v$ are behavioral strategies of the game described in section 2.1.

[5] For interesting recent extensions to heterogeneous beliefs, see Carlier and Renou [1] and [2].

[6] All results extend to the case where $v$ is any non-negative payment.

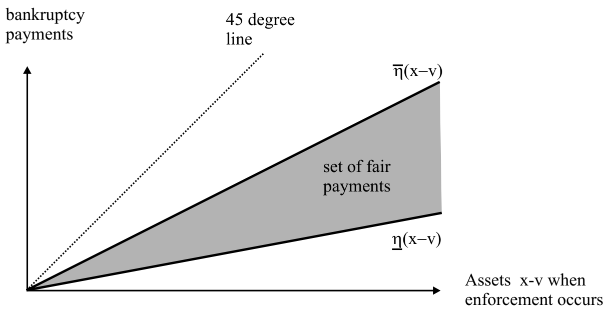


Figure 1: Feasible Bankruptcy Payments

We require that $v_x$, $e_v$, $\beta$ and $\beta'_v$ are a perfect Bayesian Nash equilibrium (PBNE), which is defined as follows.

**Definition 1** *$\{v_x, e_v, \beta, \beta'_v\}$ is a perfect Bayesian Nash equilibrium (PBNE) in behavioral strategies for the game given by $\ell(x, v)$ if and only if*

1. *$v_x$ solves $\max_{v_x} x - \sum_{v \in V} \big(v + e_v \ell(x, v)\big) v_x(v)$;*
2. *$e_v$ solves $\max_{e_v} v + \sum_{x \in X} e_v \big(\ell(x, v) - c\big) \beta'_v(x)$;*
3. *$\beta'_v(x) = \frac{v_x(v)\beta(x)}{\sum_{y \in X} v_y(v)\beta(y)}$ if the denominator is non-zero. Otherwise, $\beta_v$ is an arbitrary probability on $\{x \in X | x \geq v\}$.*

An important feature implied by a PBNE is that it builds in "incentive compatibility." By this we mean that the entrepreneur's payment $v$ is chosen optimally via condition 1 of definition 1. As we shall see, it is not optimal for the entrepreneur to reveal all information. The reason is that condition 2 of definition 1 specifies that the investor's enforcement choice must also be optimal. If all information were revealed, the investor would have no incentive to enforce at $t = 2$. This, in turn, would imply that the entrepreneur would not reveal information truthfully, but would instead announce the realization that results in the least payment.

## 2.3 Fair Enforcement

The court's role in this model is to enforce payments, when requested. Rather than mechanically enforcing any payment that is specified, courts typically are concerned with the ex-post fairness of the outcome. One way to impose fairness is to endow the court with a concave utility function $u(\pi_E, \pi_I)$, where $\pi_E$ is the entrepreneur's payment and $\pi_I$ is the investor's payment. We require $u$ to be homothetic. Thus, fairness is not linked to the total surplus available for distribution. If the $t = 1$ payment is $v$ and enforcement is requested, then the court solves $\max_{\pi_E,\pi_I} u(\pi_E, \pi_I)$ s.t. $\pi_E + \pi_I = x - v$. The solution of this problem is of the form $\pi_I = \eta(x-v)$, where $\underline{\eta} \le \eta \le \bar{\eta}$, i.e., the payment is in the cone, as indicated in figure 1.[7] This follows from the fact that when preferences are homothetic, the wealth expansion path is a straight line for strictly concave preferences and a cone for concave preferences. When the wealth expansion path is a straight line, contracting parties have no discretion in setting payments because $\ell(x, v) = \eta(x-v)$. However, in real life we observe a reasonable amount of discretion, which is consistent with preferences that are not strictly concave. Figure 1 shows that by an appropriate choice of $\ell$, any payment between $\bar{\eta}(x-v)$ and $\underline{\eta}(x-v)$ in the cone can be obtained. The only case in which the payment need not be in the fair cone is if the investor has given up all claims against the entrepreneur and the court therefore has no authority to enforce the contract, i.e., if $\ell(\cdot, v) \equiv 0$.

This leads to the following definition.

**Definition 2** *Payment function $\ell$ is fair if and only if either $\underline{\eta}(x-v) \le \ell(x, v) \le \bar{\eta}(x-v)$, for all $x \in X$ or $\ell(\cdot, v) \equiv 0$.*

Throughout the paper, we assume that $0 < \underline{\eta} \le \bar{\eta} < 1$.

### 2.4 The Coalitional Perfect Bayesian Nash Equilibrium

We now extend the definition of a PBNE to allow for coalitional deviations. As discussed in Krasa and Villamil [12], in a standard (not coalitional) PBNE it may be mutually beneficial for agents to alter the contract. If this occurs at a stage at which some information has been revealed, such a deviation is commonly referred to as renegotiation. Krasa and Villamil [12] use a non-cooperative framework and impose a renegotiation proofness constraint on the set of solutions. In contrast, we adopt a framework which has both cooperative and non-cooperative elements. The cooperative solution concept that we introduce solely imposes that agents take advantage of *mutually beneficial* deviations that are *common knowledge*, and does not depend on the order of moves in the game.[8] At $t = 0$ the contract choice is individually rational and ex-ante efficient (i.e., cooperative), and at $t = 1$ and $t = 2$ agents can cooperatively change the continuation contract. In addition, at time $t = 1$ payment $v$ is chosen by strategy $v_x$ as part of a non-cooperative PBNE, and at $t = 2$ enforcement choice $e_v$ is non-cooperative (see conditions 1 and 2 in definition 1).

[7] The cone structure is important for applications of the model. For example, Krasa, Sharma and Villamil [11] use the cone to analyze how the amount of debtor/creditor protection in a legal code affects the interest rate in a loan contract and the default probability.

[8] In other words, the solution concept does not depend on the investor's bargaining power, because improvements must be mutually beneficial. Further, our approach does not impose a particular game form that induces strategic transfer of information. See footnote 10.

Coalitional deviations can occur in each of the three time periods $t = 0, 1, 2$. The possibility to deviate is foreseen by the agents, hence agents' choices are time consistent. In other words, an unimprovable contract at time $t$ is a contract that cannot be improved upon by any other time consistent contract.[9] Incorporating time consistency into the definition of improvability automatically leads to a recursive definition, which we now provide.

**Definition 3** *$\{\ell(x, v), \upsilon_x, e_v, \beta, \beta'_v\}$ is a cPBNE if and only if*

1. *$\{\upsilon_x, e_v, \beta, \beta'_v\}$ is a PBNE for the game given by $\ell(x, v)$.*

2. *$\{\ell(x, v), e_v, \beta'_v\}$ is unimprovable at $t = 2$ for any $v$ that occurs with positive probability, i.e., no $v$ exists with $\upsilon_x(v) > 0$ for some $x \in X$ and $\{\tilde{\ell}(x, v), \tilde{v}, \tilde{e}_v, \beta'_v\}$ such that*

$$x - \tilde{v} - \tilde{e}_v \ell(x, v) \geq x - v - e_v \ell(x, v) \text{ for } \beta'_v \text{ a.e. } x$$
$$\sum_{x \in X} \tilde{v} + \tilde{e}_v(\ell(x, v) - c)\beta'_v(x) \geq \sum_{x \in X} v + e_v(\ell(x, v) - c)\beta'_v(x)$$

*where at least one inequality is strict.*

3. *$\{\ell(x, v), \upsilon_x, e_v, \beta, \beta'_v\}$ is unimprovable at $t = 1$, i.e., no $\{\tilde{\ell}(x, v), \tilde{\upsilon}_x, \tilde{e}_v, \beta, \tilde{\beta}'_v\}$ exists where conditions 1 and 2 are satisfied, $\tilde{\ell}$ is fair, and both parties are better off in the exogenous interim period, i.e.,*

$$\sum_{v \in V} \big(x - v - \tilde{e}_v \tilde{\ell}(x, v)\big)\tilde{\upsilon}_x(v) \geq \sum_{v \in V} \big(x - v - e_v \ell(x, v)\big)\upsilon_x(v), \text{ for all } x \in X$$
$$\sum_{x \in X}\sum_{v \in V} \big(v + \tilde{e}_v(\tilde{\ell}(x, v) - c)\big)\tilde{\upsilon}_x(v)\beta(x) \geq \sum_{x \in X}\sum_{v \in V} \big(v + e_v(\ell(x, v) - c)\big)\upsilon_x(v)\beta(x).$$

*where at least one inequality is strict.*

4. *$\{\ell(x, v), \upsilon_x, e_v, \beta, \beta'_v\}$ is unimprovable ex-ante, i.e., no $\{\tilde{\ell}(x, v), \tilde{\upsilon}_x, \tilde{e}_v, \beta, \tilde{\beta}'_v\}$ exists where conditions 1, 2, and 3 are satisfied, $\tilde{\ell}$ is fair, and both parties are better off ex ante, i.e.,*

$$\sum_{x \in X}\sum_{v \in V} \big(x - v - \tilde{e}_v \tilde{\ell}(x, v)\big)\tilde{\upsilon}_x(v)\beta(x) \geq \sum_{x \in X}\sum_{v \in V} \big(x - v - e_v \ell(x, v)\big)\upsilon_x(v)\beta(x)$$
$$\sum_{x \in X}\sum_{v \in V} \big(v + \tilde{e}_v(\tilde{\ell}(x, v) - c)\big)\tilde{\upsilon}_x(v)\beta(x) \geq \sum_{x \in X}\sum_{v \in V} \big(v + e_v(\ell(x, v) - c)\big)\upsilon_x(v)\beta(x).$$

*where at least one inequality is strict.*

[9] A contract which is unimprovable does not admit a mutually beneficial deviation.

Conditions 2, 3, and 4 of definition 3 consider coalitional deviations in each period:

Condition 2 specifies that in the final period, agents cannot improve upon the status quo given their updated information. The entrepreneur knows $x$ and the investor's information is summarized by the updated prior $\beta'_v$. A coalitional deviation is possible if the alternative contract makes the investor better off with respect to belief $\beta'_v$ and if the entrepreneur is better off in all states $x$ that cannot be excluded given payment $v$, i.e., all states $x$ with $\beta'_v(x) > 0$.[10] Also, if a mutually beneficial deviation $\{\tilde{\ell}(x, v), \tilde{v}, \tilde{e}_v, \beta'_v\}$ exists at $t = 2$, then one also exists in which $\tilde{e}_v = 0$.

Condition 3 specifies that no coalitional improvement exists at $t = 1$ when the entrepreneur knows realization $x$, but no information has yet been revealed to the investor. Because any mutually beneficial deviation at $t = 1$ must fulfill conditions 1 and 2, the entrepreneur's payment $v$ and the enforcement decision must be optimal. Moreover, the deviation at $t = 1$ should not admit a further deviation at $t = 2$. The fact that future deviations are foreseen is crucial in order to get existence of equilibria in our model.

Finally, condition 4 is similar to condition 3 except that it uses ex-ante instead of interim expected utility. It is easy to see that condition 4 is more stringent than condition 3, i.e., any deviation that makes agents better off at the interim, also makes agents better off ex-ante.[11]

## 3 Characterization of Equilibria

The arguments proceed as follows. Lemma 1 shows that voluntary payments in any contract associated with a cPBNE must be lower than all possible realizations of $x$. If not, for the relevant realizations of $x$ the firm could improve its outcome by reducing the voluntary payment. Lemma 2 shows that a contract is unimprovable at $t = 2$ if and only if the investor's expected gain from enforcement must be at least the minimum that can be enforced on the firm. If the investor's net gain from enforcement were always less than the entrepreneur's loss, it would be mutually advantageous to deviate. The entrepreneur could bribe the investor with a higher payment to forestall enforcement. We next show that at a cPBNE enforcement is deterministic rather than stochastic and satisfies a reservation value property: Voluntary payments below some threshold $\bar{v}$ trigger enforcement with probability 1, and those above lead to enforcement with probability 0. Lemma 3 then shows that the entrepreneur's best response is to select one of two voluntary payments 0 or

[10] We do not consider alternative contracts that improve the entrepreneur only in some state $x$ but make him worse off in a state $y$ with $\beta'(y) > 0$ (in our model coalitional improvement must be common knowledge). In order to consider deviations that improve only in some states, some communication would be necessary between the parties in order to adopt the alternative contract (e.g., voting). This would require an additional information set in the game tree.

[11] This reflects the Holmström and Myerson [7] classic result that ex-ante efficiency implies interim efficiency.

$\bar{v}$. Lemma 4 shows that it may be optimal to reduce payments in the lowest state to prevent the entrepreneur from trying to bribe the investor to forgo enforcement.

**Lemma 1** *Let $\{\ell(x,v), v_x, e_v, \beta, \beta'_x\}$ be a cPBNE. Then $v < x$ for all $x \in X$ and $v \in V$ with $v_x(v) > 0$.*

**Proof.** Suppose by way of contradiction that $v \geq x$ for some $x$ and that the entrepreneur pays $v$ in state $x$. By feasibility $v = x$. The entrepreneur's payoff is therefore zero. Instead, if the entrepreneur were to pay $x - \varepsilon$, where $\varepsilon > 0$ and small, then the payoff would be at least $\underline{\eta}\varepsilon$. Because $\underline{\eta} > 0$, the payoff would be strictly increased. This contradicts condition 1 of definition 1. ■

**Lemma 2** *Let $\{v_x, e_v, \beta, \beta'_v\}$ be a PBNE. Then $\{\ell(x,v), e_v, \beta'_v\}$ does not admit a mutually beneficial deviation at $t = 2$ for any $v$ with $e_v > 0$ if and only if*

$$\sum_{x \in X} \ell(x,v)\beta'_v(x) - c \geq \min_{x \in X, \beta'_v(x) > 0} \ell(x,v). \tag{1}$$

**Proof.** We first prove sufficiency. Assume that (1) holds but that $\{\ell(x,v), e_v, \beta'_v\}$ admits a mutually beneficial deviation at $t = 2$. If $e_v = 0$ for all $v$, then it is easy to see that there is no mutually beneficial deviation. It must be the case that there exists $v$ such that $e_v > 0$.

We now show that $e_v = 1$. The investor's continuation payoff from enforcement is given by the left-hand side of (1). Next $\underline{\eta}(x - v) > 0$, by Lemma 1, and the right-hand side of (1) is strictly positive. Therefore $e_v = 1$.

Because the contract admits a mutually beneficial deviation at $t = 2$, there exists $\tilde{v}$ such that

$$\tilde{v} \leq v + \ell(x,v), \ \forall x \colon \beta'_v(x) > 0.$$

Otherwise the entrepreneur would be worse off in one of the states. Therefore,

$$\tilde{v} \leq v + \min_{x \in X, \, \beta'_v(x) > 0} \ell(x,v). \tag{2}$$

Further, the investor is strictly better off, i.e.,

$$v + \sum_{x \in X} (\ell(x,v) - c)\beta'_v(x) < \tilde{v} \tag{3}$$

Inequalities (2) and (3) contradict (1). This completes the sufficiency part of the argument.

We now prove that (1) is necessary. Assume by way of contradiction that there exists $\{\ell(x,v), e_v, \beta'_v\}$, which does not allow a coalitional deviation at $t = 2$ but violates (1) for some $v$. Let

$$\tilde{v} = v + e_v \left[ \min_{x \in X, \, \beta'_v(x) > 0} \ell(x,v) \right].$$

Clearly the investor is strictly better off as $e_v > 0$. The entrepreneur is (weakly) better off. Therefore $\{\ell(x, v), e_v, \beta'_v\}$ admits a mutually beneficial deviation at $t = 2$, a contradiction. ■

Lemma 2 implies that enforcement is deterministic.

**Corollary 1** *Let $\{\ell(x, v), v_x, e_v, \beta, \beta'_v\}$ be a cPBNE. Then $e_v$ is either 0 or 1 for all $v$ that occur with positive probability, i.e., $v_x(v) > 0$ for some $x \in X$.*

**Proof.** Assume that $e_v > 0$. Lemma 1 implies that the right-hand side of (2) is strictly positive. Therefore, the investor's payoff from enforcement is strictly positive. In contrast the payoff from not enforcing is 0. Therefore, $e_v = 1$. ■

**Lemma 3** *Let $\{\ell(x, v), v_x, e_v, \beta, \beta'_v\}$ be a cPBNE. Then there exists an alternative payoff equivalent cPBNE $\{\tilde{\ell}(x, v), \tilde{v}_x, \tilde{e}_v, \beta, \tilde{\beta}'_v\}$ such that the following properties are satisfied:*

1. *At most the two payments $0$ and $\bar{v}$ occur with positive probability, i.e, $\tilde{v}_x(v) = 1$ only if either $v = 0$ or $v = \bar{v}$.*
2. *Enforcement takes place only if $v < \bar{v}$, i.e., $\tilde{e}_v = 1$ if $v < \bar{v}$ and $\tilde{e}_v = 0$ if $v \geq \bar{v}$.*

**Proof.** We first prove 2. Let $\hat{v}$ be some payment which occurs with positive probability from an ex ante perspective and which is followed by no enforcement, i.e., $v_x(\hat{v}) > 0$ for some $x \in X$ and $e_{\hat{v}} = 0$. Condition 1 of definition 1 then implies that $v_x = 0$ for all $v > \hat{v}$ irrespective of the value of $e_v$. Hence without loss of payoff for any agent we can set $e_v = 0$ for all $v > \hat{v}$. If $\hat{v} = 0$, then the lemma is true. So let $\hat{v} > 0$ and consider $v < \hat{v}$. By Corollary 1, $e_v$ is either 0 or 1. If $e_v = 0$, then condition 1 of definition 1 implies that $v_x(\hat{v}) = 0$, a contradiction. So, $e_v = 1$. Let $\bar{v} = \hat{v}$. Thus, there is at most one payment $\bar{v} > 0$ such that $v_x(\bar{v}) > 0$ and $e_{\bar{v}} = 0$.

We now prove 1. If $\bar{v} = 0$, the lemma is true, so let $\bar{v} > 0$. Consider all payments $v < \bar{v}$ and redefine for each $x$

$$\tilde{\ell}(x, 0) = v + \ell(x, v) \text{ for some } v \text{ such that } v_x(v) > 0; \tag{4}$$

$$\tilde{\ell}(x, 0) = \max \ell(x, v) \text{ if } v_x(v) = 0 \text{ for all } v < \bar{v}; \tag{5}$$

$$\tilde{v}_x(0) = \sum_{v<\bar{v}} v_x(v). \tag{6}$$

Because of statement 2 of the lemma and corollary 1, $\tilde{e}_0 = 1$. Since for all $v < \bar{v}$ and $\tilde{v} < \bar{v}$ over which the entrepreneur is indifferent we have $v + \ell(x, v) = \tilde{v} + \ell(x, \tilde{v}) = \tilde{\ell}(x, 0)$, it follows that (4) and (5) are well defined and the payoffs of the lender and entrepreneur are not affected by the reconstruction of the contract. Hence, $\tilde{v}_x(0)$ is optimal.

To show that $\tilde{\ell}(x,0)$ is feasible, note that optimality implies that for all $v < \bar{v}$ and $x$ such that $v_x(v) > 0$ we have $v + \ell(x,v) \leq \bar{\eta}x$. By definition $\underline{\eta}(x - v) \leq \ell(x,v)$. This implies $\underline{\eta}x \leq v + \ell(x,v)$. Hence, $\underline{\eta}x \leq \tilde{\ell}(x,0) \leq \bar{\eta}x$.

We now prove that $\{\tilde{v}_x, \tilde{e}_v, \beta, \tilde{\beta}'_v\}$ does not admit a mutually beneficial deviation at $t = 2$. Because the original contract $\{v_x, e_v, \beta, \beta'_v\}$ does not admit a mutually beneficial deviation, condition 1 of lemma 2 must hold. This, and 3 of definition 1 imply

$$\sum_{x\in X} \ell(x,v)v_x(v)\beta(x) \geq \left[\min_{x\in X,\beta'_v(x)>0} \ell(x,v) + c\right] \sum_{x\in X} v_x(v)\beta(x),$$

for all $v$ such that $v_x(v) > 0$ and $v < \bar{v}$. Therefore

$$\sum_{x\in X}[v + \ell(x,v)]v_x(v)\beta(x) \geq \left[v + \min_{x\in X,\beta'_v(x)>0} \ell(x,v) + c\right] \sum_{x\in X} v_x(v)\beta(x).$$

This implies

$$\sum_{v<\bar{v},v_x(v)>0} \sum_{x\in X}\Big[v+\ell(x,v)\Big]v_x(v)\beta(x) \geq \sum_{v<\bar{v},v_x(v)>0} \left[v+\min_{x\in X,\beta'_v(x)>0} \ell(x,v)+c\right] \sum_{x\in X} v_x(v)\beta(x).$$

This implies

$$\sum_{x\in X} \tilde{\ell}(x,0)\tilde{v}_x(0)\beta(x) \geq \sum_{v<\bar{v},v_x(v)>0} \left[\min_{x\in X,\beta'_v(x)>0}(v + \ell(x,v) + c)\right] \sum_{x\in X} \tilde{v}_x(0)\beta(x).$$

The right hand side equals

$$\left[\min_{x\in X,\tilde{\beta}'_0(x)>0} \tilde{\ell}(x,0) + c\right] \sum_{x\in X} \tilde{v}_x(0)\beta(x).$$

Finally, this implies

$$\sum_{x\in X} \tilde{\ell}(x,0)\tilde{\beta}'_v(0) \geq \min_{x\in X,\tilde{\beta}(0)>0} \tilde{\ell}(x,0) + c. \tag{7}$$

Therefore, Lemma 2 implies that $\{\tilde{v}_x, \tilde{e}_v, \beta, \tilde{\beta}'_v\}$ does not admit a mutually beneficial deviation at $t = 2$.

By Lemma 1 the right hand side of (7) is strictly positive and hence $e_0 = 1$ is optimal. ■

As a consequence of Lemma 3, it sufficient to consider strategies $v_x$ for which at most two payments occur with positive probability. If payment $\bar{v}$ is made, then no enforcement occurs. Thus, only payments $\ell(\cdot, 0)$ matter. Moreover, $v_x(0)$ is now the probability that the entrepreneur does not pay, i.e., the default probability.

We will use the following assumption to simplify the analysis of the game.

**Assumption 1** $\beta(\underline{x}) \leq \beta(x),\ \forall x \leq \frac{\bar{\eta}}{\underline{\eta}}\underline{x}$.

If $\underline{x}$ is close to 0, then assumption 1 imposes no restriction on the probability distribution. If the probabilities $\beta(x)$ are increasing, then assumption 1 holds by default. If $\beta(x)$ increases but then decreases, as long as $\underline{\eta}$ is not too small, assumption 1 also holds.

The next lemma shows that the right-hand side of (1) is minimized at $x = \underline{x}$.

**Lemma 4** *Let assumption 1 be satisfied. Let $\{v_x, e_v, \beta, \beta'_v\}$ be a cPBNE that satisfies (1). Then there exists an alternative cPBNE $\{\tilde{v}_x, \tilde{e}_v, \beta, \tilde{\beta}'_v\}$ which gives the same expected payoffs to the investor and the entrepreneur, and for which $\underline{x}$ solves* $\min_{x \in X, \tilde{\beta}'_v(x)>0} \tilde{\ell}(x, 0)$.

**Proof.** Because of Lemma 3 we can assume that only payment $v = 0$ leads to enforcement. Now assume that there exist $\hat{x} > \underline{x}$ such that $\ell(\underline{x}, 0) > \ell(\hat{x}, 0)$. Then $\ell(\underline{x}, 0) \leq \bar{\eta}\underline{x}$ and $\ell(\hat{x}, 0) \geq \underline{\eta}\hat{x}$ implies $\bar{\eta}\underline{x} \geq \underline{\eta}\hat{x}$. Assumption 1 implies that $\beta(\underline{x}) \leq \beta(\hat{x})$. Now let

$$\tilde{\ell}(x, 0) = \begin{cases} \ell(\hat{x}, 0) & \text{if } x = \underline{x}; \\ \dfrac{\beta(\underline{x})(\ell(\underline{x}, 0) - \ell(\hat{x}, 0))}{\beta(\hat{x})} + \ell(\hat{x}, 0) & \text{if } x = \hat{x}; \\ \ell(x, 0) & \text{otherwise.} \end{cases}$$

Note that $\tilde{\ell}(x, 0)$ is in the fair cone. For $x \neq \underline{x}, \hat{x}$ this is immediate. Now let $x = \underline{x}$. Then $\tilde{\ell}(\underline{x}, 0) = \ell(\hat{x}, 0) < \ell(\underline{x}, 0) \leq \bar{\eta}\underline{x}$ and $\tilde{\ell}(\underline{x}, 0) = \ell(\hat{x}, 0) \geq \underline{\eta}\hat{x} > \underline{\eta}\underline{x}$. Therefore, $\tilde{\ell}(\underline{x}, 0)$ is in the fair cone. Finally, let $x = \hat{x}$. Then $\tilde{\ell}(\hat{x}, 0) = \frac{\beta(\underline{x})}{\beta(\hat{x})}(\ell(\underline{x}, 0) - \ell(\hat{x}, 0)) + \ell(\hat{x}, 0) < \ell(\underline{x}, 0) - \ell(\hat{x}, 0) + \ell(\hat{x}, 0) = \ell(\underline{x}, 0) \leq \bar{\eta}\underline{x} < \bar{\eta}\hat{x}$. Similarly, $\tilde{\ell}(\hat{x}, 0) = \frac{\beta(\underline{x})}{\beta(\hat{x})}(\ell(\underline{x}, 0) - \ell(\hat{x}, 0)) + \ell(\hat{x}, 0) > \ell(\hat{x}, 0) \geq \underline{\eta}\hat{x}$. Therefore, $\tilde{\ell}(\hat{x}, 0)$ is also in the fair cone.

Next, note that $\beta(\underline{x})\ell(\underline{x}, 0) + \beta(\hat{x})\ell(\hat{x}, 0) = \beta(\underline{x})\tilde{\ell}(\underline{x}, 0) + \beta(\hat{x})\tilde{\ell}(\hat{x}, 0)$. Therefore, the expected payments are the same under $\ell(x, 0)$ and $\tilde{\ell}(x, 0)$. Similarly, the left-hand side of (1) does not change if we replace $\ell$ by $\tilde{\ell}$. Since $\tilde{\ell}(\underline{x}, 0) < \ell(\hat{x}, 0)$ it follows immediately that the right-hand side of (1) is minimized at $x = \underline{x}$ if we replace $\ell$ by $\tilde{\ell}$. ■

# 4 The Equilibrium Contract Problem

Theorem 1 below shows that equilibria of our model correspond to solutions of the following optimization problem.

**Problem 1** *At $t = 0$, choose $\{\bar{v}, \ell(x, 0), v_x(0)\}$ to maximize*

$$E_0[u_I(x)] = \sum_{x \in X} \Big[\big(\ell(x, 0) - c\big)v_x(0) + \bar{v}(1 - v_x(0))\Big]\beta(x) \tag{8}$$

*Subject to*

$$E_0[u_F(x)] = \sum_{x\in X}\Big[x - \ell(x,0)v_x(0) - \bar{v}(1 - v_x(0))\Big]\beta(x) \geq \bar{u}_F \tag{9}$$

$$v_x(0) = \begin{cases} 1 & \text{if } \bar{v} > \ell(x,0) \\ 0 & \text{if } \bar{v} < \ell(x,0) \\ \alpha \in [0,1] & \text{if } \bar{v} = \ell(x,0) \end{cases} \tag{10}$$

*If $v_x(0) > 0$ for an $x \in X$ (i.e., bankruptcy occurs) then:*

$$\sum_{x\in X}\ell(x,0)\beta_0'(x) - c \geq \ell(\underline{x},0) \tag{11}$$

$$\bar{\eta}(\bar{x} - \bar{v}) - c \geq 0 \tag{12}$$

$$\underline{\eta}x \leq \ell(x,0) \leq \bar{\eta}x,\ \forall x \in X \tag{13}$$

(8) is the investor's ex-ante expected payoff. Given realization $x$, the entrepreneur defaults with probability $v_x(0)$. In equilibrium, the investor pays cost $c$ to request enforcement when default occurs. Otherwise, with probability $1 - v_x(0)$ the entrepreneur pays $\bar{v}$. The left-hand side of (9) is the entrepreneur's expected payoff. By fixing the entrepreneur's expected utility at a level $\bar{u}_F$ and maximizing the investor's payoff we obtain ex-ante efficient allocations. (10) requires the entrepreneur's bankruptcy choice $v_x(0)$ to maximize his expected payoff. If $\bar{v}$ exceeds the payment from enforcement, $\ell(x,0)$, the entrepreneur chooses to default (i.e., not pay $\bar{v}$). If the inequality is reversed, the entrepreneur chooses to pay $\bar{v}$ with probability 1. If the payments are the same, then the entrepreneur can randomize with probability $\alpha$. (11) rules out mutually beneficial deviations. (12) ensures that it is not beneficial for the entrepreneur to make a payment $0 < v < \bar{v}$.[12] (13) ensures that payments are in the fair cone.

**Theorem 1** *Let $\{\ell(x,v), v_x, e_v, \beta, \beta_v'\}$ be a cPBNE. Let $\bar{u}_F$ be the entrepreneur's ex-ante expected utility. Then the solution of problem 1 provides the same expected payoffs to the investor and entrepreneur in each state $x$.*

*Conversely, let $\bar{u}_F \geq 0$ and let $\{\ell(x,v), v_x(0), \bar{v}\}$ be a solution to problem 1 which, if there is more than one, gives the highest expected payoff to the investor. If the entrepreneur's expected payoff is non-negative then $\{\ell(x,v), v_x(0), \bar{v}\}$ can be supported as a cPBNE.*

[12]Condition 3 of definition 3 requires that agents cannot make any improvements ex-ante. By definition such improvements include coordinating beliefs $\beta_v'$ to support the highest possible ex-ante utilities. This is achieved by the following "optimistic beliefs:" If a deviation $0 < v < \bar{v}$ were to occur, the investor believes that the highest possible $\bar{x}$ was realized.

**Proof.** Lemma 2 and lemma 4 prove that mutually beneficial deviations do not exist if and only if (11) holds. Lemma 3 implies that we can focus on equilibria where only payments 0 and $\bar{v}$ occur. Given such an equilibrium, (10) is equivalent to condition 1 of definition 1. Condition 2 is implied by (11) and (12). In particular, (11) implies that $\sum_{x\in X} \ell(x,0)\beta_0'(x) - c \geq \ell(\underline{x},0) > 0$. Therefore, the investor will request enforcement when $v = 0$. For off equilibrium path payments $0 < v < \bar{v}$, (11) is the weakest restriction that ensures that enforcement takes place, thereby ruling out such payments. Because of condition 2 of definition 1, in order for $e_v = 1$ we must have

$$\sum_{x\in X}(\ell(x,v) - c)\beta_v'(x) \geq 0. \tag{14}$$

First, note that we can increase the left-hand side of (14) by setting $\ell(x,v) = x - v$ for $0 < v < \bar{v}$, as this does not affect payments on the equilibrium path. Second, we can assume that $\beta_v'$ puts all the mass on $\bar{x}$, the highest possible realization, which satisfies condition 3 of definition 1. Inserting $\ell(x,v) = x - v$ and $\beta_v'(\bar{x}) = 1$ into (14) immediately implies (12). The second statement of the theorem is now immediate.

Finally, let $\bar{u}_F$ be the entrepreneur's expected utility. Consider a cPBNE but assume by contradiction that there exists an alternative contract that increases the objective of problem 1 and satisfies all the constraints. Because the alternative contract satisfies all constraints, it is a PBNE and does not admit a mutually beneficial deviation. The alternative contract gives at least utility $\bar{u}_F$ to the entrepreneur and strictly increases the investor's expected utility. This contradicts condition 3 of definition 3. ■

We can now prove that cPBNEs exist for our model.

**Theorem 2** *cPBNEs exist. In equilibrium, the project will receive funding if costs $c$ are not too high, and $\bar{\eta}$ and the project's expected return are not too low.*

**Proof.** First, assume that any $\{\ell(x,0), v_x(0), \bar{v}\}$ that satisfies the constraints of problem 1 for $\bar{u}_F = 0$ gives the entrepreneur a negative payoff. Then the only equilibrium is autarky.

Now assume that it is possible to satisfy the constraints of problem 1 and give the entrepreneur a non-negative payoff. Note that this is always the case if $\bar{\eta}$ is close to 1, $\sum_{x\in X} x\beta(x) > 1$ and $c$ is small. Fix $v_x(0)$. The set of payment functions $\ell(x,0)$ is finite dimensional because $X$ is finite. The objective of problem 1 and the constraints are therefore continuous in $\ell(x,0)$ and $\bar{v}$. Because the constraint set is bounded, a maximum exists, and the set of solutions to problem 1 keeping $v_x(0)$ fixed is compact. Therefore, there exists a solution that maximizes the investor's payoff. Because there is only a finite number of different choices of $v_x(0)$, problem 1 has a solution. Among the solutions there exists one that maximizes the investor's payoff. Theorem 1 therefore implies the result. ■

## 5 Equilibrium Contracts

We now characterize solutions of problem 1. We first define a debt contract. The key feature of debt is that default occurs only for "low" project realizations.

**Definition 4** *$\{\ell(x,0), \bar{v}, v_x(0)\}$ is a **debt contract** if the set of all default states $\mathcal{D} = \{x | v_x(0) > 0\}$ is an interval of low realizations.*

*The contract is **simple debt** if $\ell(x,0) = \bar{\eta}x$ for all $x$ (i.e., the ex-ante contract specifies that all non-exempt assets are seized in bankruptcy).*

The model accommodates both inability to pay and willful default (cf., Krasa, Sharma and Villamil [11]).

**Definition 5** *Let $\mathcal{D} = \{x | v_x(0) > 0\}$ be the set of bankruptcy states. Then*

1. *$\mathcal{D}_a = \{x \in \mathcal{D} | x < \bar{v}\}$ is the set of default states where the entrepreneur is **unable** to pay.*[13]

2. *$\mathcal{D}_w = \{x \in \mathcal{D} | x \geq \bar{v}\}$ is the set of default states where the entrepreneur is **unwilling** to pay.*

3. *$\mathcal{N} = X \setminus \mathcal{D}$ is the set of non-default states.*

Theorem 3 proves that a debt contract that distinguishes between ability and willingness to pay is optimal in the model. Further, when constraint (11) does not bind, a simple debt contract is optimal.[14]

**Theorem 3** *In all solutions of Problem 1, $\mathcal{D}_a$ is an interval where the entrepreneur is not able to pay and $\mathcal{D}_w$ is an interval where the entrepreneur is not willing to pay, with $\mathcal{D}_a \leq \mathcal{D}_w \leq \mathcal{N}$.*

1. *If (11) does not bind, then simple debt contracts are optimal.*

2. *If (11) binds, then debt contracts are optimal with $\ell(x,0)$ increasing in $x$ for all $x$ with $\ell(x,0) < \bar{v}$ (i.e., for all realizations $x$ where the entrepreneur strictly prefers default).*

---

[13] If $x < \bar{v}$, then by feasibility default must occur. Thus, $\mathcal{D}_a$ can also be defined as $\mathcal{D}_a = \{x | x < \bar{v}\}$.

[14] When (11) binds debt rather than simple debt is optimal. Sharma [15] shows that lowering $\ell(\underline{x}, 0)$ to a value strictly less than $\bar{\eta}\underline{x}$ lowers the right-hand side of (11), thereby weakening the constraint. This allows agents to find better outcomes. Intuitively, leaving the entrepreneur with some assets in the lowest state makes the entrepreneur less willing to renegotiate (i.e., deviate). This increases the investor's incentive to request enforcement rather than to renegotiate.

**Proof.** We consider each case.

*Case 1.* Constraint (11) does not bind.

Consider Problem 1 without (11). If (9) does not bind, then $\ell(x,0) = x$ for all $x$ with $v_x(0) > 0$. Otherwise, if $\ell(x,0) < x$ the investor's payoff can be increased without violating any constraint.

Now assume that constraint (9) binds. Substituting (9) into the objective of Problem 1 yields $\sum_{x\in X} x\beta(x) - \bar{u}_F - c\sum_{x\in X} v_x(0)\beta(x)$. Therefore, Problem 1 is equivalent to minimizing

$$\sum_{x\in X} v_x(0)\beta(x); \tag{15}$$

subject to (9), (10), (12) and (13), where the inequality in (9) is reversed. Now choose $\tilde{v}$ such that

$$\sum_{x<\frac{\tilde{v}}{\bar{\eta}}} \bar{\eta}x\beta(x) + \sum_{x\geq\frac{\tilde{v}}{\bar{\eta}}} \tilde{v}x\beta(x) = \bar{u}_F. \tag{16}$$

Note that $\tilde{v}$ exists and $\tilde{v} \leq \bar{v}$ since $\ell(x,0) \leq x$. Define $\tilde{v}_x(0) = 1$ if and only if $\bar{\eta}x < \tilde{v}$. Clearly, (10) is satisfied. (16) immediately implies that (9) holds. Next, $\tilde{v}_x(0) \leq v_x(0)$. Hence, (15) is decreased. Thus, choosing $\tilde{\ell}(x,0) = x$ is optimal because it minimizes expected enforcement costs.

Finally, we show that $\mathcal{D}_a$ and $\mathcal{D}_w$ are intervals. By definition, $\mathcal{D}_a$ is an interval. Because $\ell(x,0) = x$, we get $\mathcal{D}_w = \{x|\bar{\eta}x < \bar{v} \leq x\}$. Therefore, $\mathcal{D}_w$ is also an interval.

*Case 2.* Constraint (11) binds.

Let $\tilde{\mathcal{D}}_w = \{x \in \mathcal{D}_w|\ell(x,0) < \bar{v}\}$. We can redefine $\ell(x,0)$ such that it is monotonically increasing on $\mathcal{D}_a \cup \tilde{\mathcal{D}}_w$: Let $x, \tilde{x} \in \mathcal{D}_a \cup \tilde{\mathcal{D}}_w$ with $x < \tilde{x}$. Assume that $\ell(x,0) > \ell(\tilde{x},0)$. We can then find $\tilde{\ell}(x,0) = \tilde{\ell}(\tilde{x},0)$ such that $\beta(x)\tilde{\ell}(x,0) + \beta(\tilde{x})\tilde{\ell}(\tilde{x},0) = \beta(x)\ell(x,0) + \beta(\tilde{x})\ell(\tilde{x},0)$. Repeating this argument for all states, we get a monotone payoff function that yields the same expected payoff as $\ell(x,0)$. Therefore, constraints (9), (10), (12), and (13) are satisfied for $\tilde{\ell}$. Similarly, the left-hand side of (11) does not change. Clearly, (11) holds if $\tilde{\ell}(\underline{x},0) = \ell(\underline{x},0)$. Thus, assume $\tilde{\ell}(\underline{x},0) \neq \ell(\underline{x},0)$. This implies that $\ell(\underline{x},0) > \ell(\tilde{x},0)$ for some $\tilde{x} \in \mathcal{D}_a \cup \tilde{\mathcal{D}}_w$. Then the above construction yields $\tilde{\ell}(\underline{x},0) \leq \ell(\underline{x},0)$. As a consequence, (11) is also satisfied.

Finally, we show that the default regions are intervals: Clearly, $\mathcal{D}_a = \{x|x < \bar{v}\}$ is an interval. Because $\ell(x,0)$ is monotone in $x$, it follows that $\{x|\ell(x,0) < \bar{v} \leq x\}$ is an interval. Therefore, $\tilde{\mathcal{D}}_w$ is an interval. Without affecting payoffs or constraints, we can also redefine $\ell(x,0)$ and $v_x(0)$ on $\mathcal{D}_w \setminus \tilde{\mathcal{D}}_w$ such that this set becomes the interval bordering $\tilde{\mathcal{D}}_w$. ■

## 6 Concepts of Efficiency

We now clarify our notion of efficiency with those in the literature.[15] (Pareto) efficiency is often viewed as a minimal test that any welfare optimal outcome should satisfy. An allocation is said to be efficient if and only if there is no other feasible allocation that makes some agents better off without making any agent worse off. This criterion is straightforward under complete information, but in a classic paper Holmström and Myerson [7] show that under incomplete information efficiency is more difficult to define. They identify six classes of efficient mechanisms which are distinguished by two criteria – three time periods at which welfare is evaluated and two incentive cases. Holmström and Myerson note that the timing of the welfare evaluation directly affects the information conditions under which an agent's expected utility is evaluated. They identify the time periods as ex ante, interim and ex post. The two incentive cases take into account whether or not constraints which make it optimal to reveal information are considered.

*Classical efficiency* refers to the case where no incentive constraints are considered (e.g., the incentive constraint is ignored or the state becomes public information). In contrast, *incentive efficiency* refers to the situation where information is private and a constraint which governs the agents' incentive to reveal information is satisfied. The three time periods are distinguished as follows:

1. *Ex ante incentive/classical efficiency:* Ex-ante, agents have not yet received any private information, and expected utility is therefore not conditional on it. In our model at $t = 0$, $\beta(x)$ is the agents' common prior before information is revealed. If no deviation exists that both agents would agree to, then the allocation is ex-ante incentive efficient in the sense of Holmström and Myerson.

2. *Interim incentive/classical efficiency:* Expected utility is evaluated given each agent's type (i.e., private information). In our model this corresponds to the situation at $t = 1$ where the entrepreneur observes $x$, but the investor remains uninformed.

3. *Ex post classical/incentive efficiency:* Utility is evaluated at the realized state of nature, even if all agents do not know the state (which is the case in our model when enforcement does not occur). Standard Bayesian incentive compatibility is often used in connection with the definition of ex-post efficiency. In a mechanism design framework, agents simultaneously truthfully reveal all of their information to a "planner" who then uses it to design an efficient allocation. As a consequence, there is no further opportunity for agents to make strategic choices based on information (as no private information remains). In contrast, in our model the investor chooses whether or not to invoke costly enforcement at $t = 2$ based on payment $v$ observed

[15] Because there are only two agents in the model, the absence of a coalitional deviation corresponds to efficiency.

at $t = 1$, and it is not possible to reveal all information at $t = 1$ without destroying the investor's incentive to enforce. In fact, Lemma 3 indicates that information revelation will be minimal. The coalitional deviation at $t = 2$ therefore does not occur when information is complete. Thus, neither ex-post classical nor ex-post incentive efficiency is applicable in our model at $t = 2$, and we must introduce a new notion.

We call the notion that we introduce endogenous interim efficiency (i.e., the absence of coalitional deviations at $t = 2$). This concept corresponds most closely to posterior efficiency in Forges [3], which extends Green and Laffont's [5] posterior implementation approach. Forges considers a Bayesian collective-choice problem where a mechanism $\mu$ selects a decision $d \in D$, and $\nu(\delta) \in \Delta(D)$ is a feasible alternative decision where $\nu$ is a probability on $D$. Then $\mu$ is posterior efficient if it is not possible to increase the expected utility, given $d$, of *all* agents.

Let $t_i$ denote agent $i$'s type, and let $t_{-i}$ denote the type of the remaining agents. Forges defines posterior efficiency for a mechanism as follows:

**Definition 6** *Mechanism $\mu$ is posterior efficient if the following is not true:*
*There exist $d \in D$, $P_\mu(d) > 0$ and an alternative mechanism $\nu \in \Delta(D)$ such that*

$$\sum_{t_{-i}} P_\mu(t_{-i}|t_i, d)[u_i(t, d) - \sum_\delta \nu(\delta)u_i(t, \delta)] < 0,$$

*for all $i = 1, ..., n$ and for all $t_i$ with $P_\mu(t_i|d) > 0$.*

No further information revelation is necessary to use $\delta$. Forge notes that this concept is appropriate when agents can communicate only through the mechanism.[16] In our case agents communicate both within the mechanism, as in Forges, and outside the mechanism because agents can make arbitrary payments $v \geq 0$ which reveal information.[17]

Forges' definition maps into condition 2 of definition 3. The choices $d \in D$ correspond to payments $v \in V$. The entrepreneur's type $t_E$ is given by realization $x \in X$. The investor has no private information, hence $t_I$ is trivial. Therefore $P_\mu(t_{-i}|t_i, d) = \beta_v'(x)$ if $t_{-i} = x = t_E$ and $t_i = t_I$, and $P_\mu(t_{-i}|t_i, d)$ assigns probability 1 to $x$ if $t_{-i} = t_I$ and $t_i = x = t_E$. $\nu(\delta)$ is the probability that alternative decision $\delta \in D$ is chosen. In our case, we could consider lotteries over $V$. However, because both agents' utilities are linear in $v$, we can consider arbitrary payments $\tilde{v} \geq 0$ instead of lotteries over $V$. Therefore our

[16]Because we assume that communication can occur only in the game, any coalitional improvements must be common knowledge. For example, this is captured, in the definitions of ex-ante and interim efficiency of Holmström and Myerson [7] or definition 5.3.5 of Hahn and Yannelis [6]. In contrast, definition 5.3.4 or the example in Proposition 5.4.6 in Hahn and Yannelis [6] capture situations where improvements are not common knowledge (i.e., the alternative decision rule improves upon the status quo in all states $\omega \in A$, but all agents need not know whether event $A$ has occurred).

[17]In other words, a planner cannot restrict a priori that agents announce only 0 or $\bar{v}$.

notion of absence of coalitional deviations at $t = 2$ in definition 3 corresponds exactly to conditions 2 and 3 of definition 6 from Forges. Condition 1 of definition 6 corresponds to our requirement that $\tilde{v}_x(v) > 0$.

Finally, recall from section 2.4 that we must also account for potential deviations at $t = 0$ and $t = 1$, and that we require $\{\ell(x, v), v_x, e_v, \beta, \beta'_v\}$ to be a PBNE. As a consequence, we need a broader notion of equilibrium. Thus in our definition 3, condition 2 corresponds to Forges, but we also require conditions 1, 3 and 4 which account for the PBNE and the possibility for deviations in previous time periods.

# 7 Concluding Remarks

This paper develops the notion of a coalitional perfect Bayesian Nash equilibrium in which the chosen strategies are optimal for a given enforceable contract, and there are no mutually beneficial deviations from the contract at any stage in the game. The possibility of mutually beneficial deviations (or improvements) is natural in financial contracting problems where higher voluntary payments can potentially reduce the likelihood of enforcement, thereby allowing agents to share the surplus generated by economizing on enforcement costs. The solution concept captures both the non-cooperative aspect of firm liquidation and the cooperative aspect of firm restructuring. The Theorems establish that equilibria of the model correspond to solutions of an optimal contracting problem that exist and resemble debt. These results are useful because they make computation and policy evaluation possible in the contract problem.

## References


[1] G. Carlier and L. Renou. A costly state verification model with diversity of opinions. forthcoming *Economic Theory*, 2003.

[2] G. Carlier and L. Renou. Existence and monotonicity of optimal debt contracts in costly state verification models. Working Paper, European University, Florence, 2003.

[3] F. Forges. Posterior efficiency. *Games and Economic Behavior*, 6:238–261, 1994.

[4] D. Gale and M. Hellwig. Incentive-compatible debt contracts: The one period problem. *Review of Economic Studies*, 52:647–663, 1985.

[5] J. Green and J. J. Laffont. Posterior implementability in a 2-person decision problem. *Econometrica*, 55:69–94, 1987.

[6] G. Hahn and N. C. Yannelis. Efficiency and incentive compatibility in differential information economies. *Economic Theory*, 10:383–411, 1997.

[7] B. Holmstrom and R. Myerson. Efficient and durable decision rules with incomplete information. *Econometrica*, 51:1799–1819, 1983.

[8] T. Ichiishi and M. Sertel. Cooperative interim contract and re-contract: Chandler's m-form firm. *Economic Theory*, 11:523–543, 1998.

[9] L. Koutsougeras and N. C. Yannelis. Incentive compatibility and information superiority of the core of an economy with differential information. *Economic Theory*, 3:195–216, 1993.

[10] L. Koutsougeras and N. C. Yannelis. Bounded rational learning in differential information economies: Core and value. *Journal of Mathematical Economics*, 31:373–391, 1999.

[11] S. Krasa, T. Sharma, and A. Villamil. The effect of enforcement on firm finance. working paper, 2003.

[12] S. Krasa and A. Villamil. Optimal contracts when enforcement is a decision variable. *Econometrica*, 68:119–134, 2000.

[13] S. Krasa and N. C. Yannelis. The value allocation of an economy with differential information. *Econometrica*, 62:881–900, 1994.

[14] S. Krasa and N. C. Yannelis. Existence and properties of a value allocation for an economy with differential information. *Journal of Mathematical Economics*, 25:165–179, 1996.

[15] T. Sharma. Optimal contracts when enforcement is a decision variable: A comment. *Econometrica*, 71:387–390, 2003.

[16] S. Williamson. Costly monitoring, financial intermediation, and equilibrium credit rationing. *Journal of Monetary Economics*, 18:159–179, 1986.

[17] N. C. Yannelis. The core of an economy with differential information. *Economic Theory*, 1:183–198, 1991.