The Gini Coefficient and Poverty Indexes: Some Reconciliations

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Poverty is usually defined as the extent to which individuals in a society or community fall below a minimal acceptable standard of living. An index of poverty is generally based on the proportion (α) of the poor people and their income distribution through the income gap ratio (β) (i.e., the average income gap of the poor people from the poverty line ω, taken as a ratio to ω itself) and other measures of income inequalities. In this context, the well-known Gini coefficient of income inequality plays a vital role. In view of the fact that the poverty indexes all relate to the income pattern of the poor, interpreted in some way or other, usually a censored (G_ω^c) or truncated (G_α) version of the classical Gini coefficient (G) is incorporated in the formulation of such indexes. Among the various poverty indexes, $\pi_A = \alpha\beta$, $\pi_T = G_\omega^c$, and $\pi_S = \alpha\{\beta + (1 - \beta)G_\alpha\}$ have been used more extensively than the others. Although each of π_S and π_T is justified on the grounds of certain plausible axioms, from a statistical point of view, generally, for smaller values of α and β, π_T is somewhat more conservative whereas π_S is more anticonservative than they should have been ideally. This phenomenon is mainly due to the two forms of the Gini coefficient G_ω^c and G_α, which behave rather differently with the variation of α, β, and the income inequalities among the poor people. This calls for a more intensive study of the behavior of the Gini coefficient under various patterns of income inequality and its role in the formulation of poverty indexes. This examination leads to consideration of a more robust version of π_S, namely, $\pi^* = \alpha\beta^{1-G_\alpha}$. In this context, TTT transformations (usually arising in reliability theory and life testing problems) are incorporated to provide a new interpretation of the Gini coefficient; in light of this, the relationship between G_α and G_ω^c and suitable bounds for either of these indexes are studied in detail. These results, in turn, facilitate the study of the relative and absolute interpretations of these poverty indexes, in the light of which the weakness of the index π_T and the relative strengths of π_S and π^* have also been discussed.

KEY WORDS: Censored and truncated Gini coefficient; Income gap ratio; Income inequality; Scaled TTT transformation; Sen and Takayama poverty indexes.

1. INTRODUCTION

An *index of poverty* is usually based on the *income distribution* $F(y)$, $y \in R^+ = [0, \infty)$, and a set *poverty line* ω (>0), so

$$\alpha = F(\omega), \quad \text{the proportion of people below the poverty line } \omega, \tag{1.1}$$

may be taken as a *crude index*. The *income gap ratio* (β) of the poor people may then be defined by

$$\beta = 1 - \omega^{-1}\left\{\alpha^{-1}\int_0^\omega y\, dF(y)\right\}. \tag{1.2}$$

Generally, both α and β are taken into consideration in the formulation of a meaningful poverty index. Indeed, from one normative standpoint of view, Sen (1976) suggested the simple poverty index

$$\pi_A = \alpha\beta, \tag{1.3}$$

although his refined index is

$$\pi_S = \alpha\{\beta + (1 - \beta)G_\alpha\}, \tag{1.4}$$

where G_α is the *Gini coefficient* of the income distribution among the poor (to be defined later). Although π_S has been used extensively, there remain some issues relating to its normative content. Takayama (1979), incorporating a somewhat different set of axioms, argued that a proper measure of the poverty is G_ω^c, the Gini coefficient of the income distribution censored at ω (as will be defined later); that is,

$$\pi_T = G_\omega^c \text{ is the Takayama poverty index.} \tag{1.5}$$

Whereas Sen (1976) advocated that an index of poverty should be given by the weighted aggregate gap of the people below the poverty line, the Takayama index is based on the income inequality of the censored income distribution truncated from above by ω (and thereby is affected by the censored probability mass $1 - \alpha$ as well). Although each of π_S and π_T is justified on the grounds of certain plausible axioms, π_T fails to satisfy a number of properties that are widely believed to be highly desirable for a poverty index [see Foster (1984) for a nice survey of these measures from an economic point of view]. Subsequent sections will show that from a statistical point of view π_T possesses some undesirable features. It will be shown later that G_α varies between 0 and β and that G_ω^c is bounded from below by αG_α and above by $\alpha\beta$. On the other hand, by definition in (1.4), π_S can never be less than $\alpha\beta$. Thus π_T is always less than or equal to π_S. Further, Sen (1976) showed that when all of the poor people have a common income, say, y_o ($\le\omega$), then $\pi_S = \alpha\beta$, irrespective of the particular value of y_o. On the other hand, π_T will be shown to be generally smaller than $\alpha\beta$, even if all of the poor people have the common income y_o when $y_o < \omega$. This apparent discrepancy between π_S and π_T may well be explained by the behavior of the Gini indexes G_α and G_ω^c under various patterns of income inequalities and the basic role played by them in the formulation of plausible poverty indexes. This examination leads us to consider a more robust version of the Sen (1976) poverty index, namely,

$$\pi^* = \alpha\{\beta^{1-G_\alpha}\}. \tag{1.6}$$

To motivate the proposed poverty index, in Section 2 I propose to have a fresh look into the Gini coefficient G in the censored and truncated cases; in this context total time on text (TTT) transformations are incorporated to provide a new interpretation of the Gini coefficient. The main results on the poverty indexes are then presented in Section

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Journal of the American Statistical Association
December 1986, Vol. 81, No. 396, Theory and Methods

3. In this respect, both the absolute and relative interpretations of the poverty indexes in (1.3)–(1.6) are considered in a statistical framework. Some general remarks are made in the concluding section.

2. THE GINI COEFFICIENT: A FRESH LOOK

First, consider the uncensored distribution F and assume that

$$\mu = EY = \int_0^\infty y\, dF(y) \quad \text{is finite.} \tag{2.1}$$

Defining the *survival function* $\bar{F}(x) = 1 - F(x),\ x \in R^+$, from (2.1) we obtain

$$\mu = \int_0^\infty \bar{F}(y)\, dy. \tag{2.2}$$

Then define

$$F^{-1}(t) = \inf\{x : F(x) \geq t\}, \qquad 0 \leq t \leq 1, \tag{2.3}$$

and

$$\xi(t) = \mu^{-1}\left\{\int_0^{F^{-1}(t)} y\, dF(y)\right\}, \qquad 0 \leq t \leq 1, \tag{2.4}$$

and consider the *Lorenz curve* depicting $\xi(t)$ as a function of t on the unit interval [0, 1] (see Fig. 1).

Then, in terms of the shaded area A, we have the Gini coefficient

$$G = 2A. \tag{2.5}$$

Analytically, G may also be expressed as

$$G = (2\mu)^{-1}E|Y_1 - Y_2|, \tag{2.6}$$

where Y_1 and Y_2 are two independent random variables, each having the same distribution F. Note that

$$Y_1 - Y_2 = (Y_1 \vee Y_2) - (Y_1 \wedge Y_2), \tag{2.7}$$

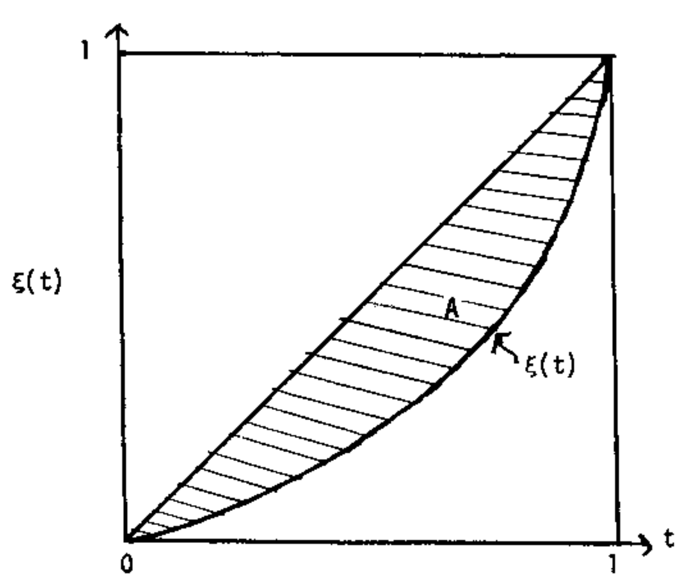

Figure 1. Lorenz Curve for a Hypothetical Income Distribution F.

where $a \vee b$ and $a \wedge b$ stand for the maximum and minimum of a and b, respectively. Therefore, by (2.6) and (2.7) we obtain

$$\begin{aligned} G &= (2\mu)^{-1}\{E(Y_1 \vee Y_2) - E(Y_1 \wedge Y_2)\} \\ &= 1 - \mu^{-1}E(Y_1 \wedge Y_2) \\ &= 1 - 2\mu^{-1}\int_0^\infty y\bar{F}(y)\, dF(y). \end{aligned} \tag{2.8}$$

At this stage I borrow the concept of TTT from the reliability theory (see Klefsjö 1983) and consider the *scaled* TTT transformation:

$$\begin{aligned} \phi(t) &= \mu^{-1}\int_0^{F^{-1}(t)} \bar{F}(u)\, du \\ &= \int_0^{F^{-1}(t)} \bar{F}(u)\, du \Big/ \int_0^\infty \bar{F}(u)\, du, \quad 0 \leq t \leq 1. \end{aligned} \tag{2.9}$$

Note that by (2.4) and (2.9)

$$\phi(t) = \xi(t) + \mu^{-1}(1 - t)F^{-1}(t), \qquad t \in [0, 1]. \tag{2.10}$$

In this context, also note that

$$TF(x) = \mu^{-1}\int_0^x \bar{F}(u)\, du, \qquad x \in R^+, \tag{2.11}$$

is called the *equilibrium renewal distribution* corresponding to F. Then

$$\begin{aligned} \bar{\phi} &= \int_0^1 \phi(t)\, dt = \int_0^\infty TF(x)\, dF(x) \\ &= \text{scaled TTT mean.} \end{aligned} \tag{2.12}$$

Note that

$$\begin{aligned} \bar{\phi} &= \mu^{-1}\int_0^\infty \left\{\int_0^x \bar{F}(u)\, du\right\} dF(x) \\ &= \mu^{-1}\left\{\iint_{0 \leq u \leq x < \infty} \bar{F}(u)\, du\, dF(x)\right\} \\ &= \mu^{-1}\left\{\int_0^\infty \bar{F}^2(u)\, du\right\} \\ &= -2\mu^{-1}\left\{\int_0^\infty u\bar{F}(u)\, d\bar{F}(u)\right\} \\ &= 2\mu^{-1}\left\{\int_0^\infty u\bar{F}(u)\, dF(u)\right\} = 1 - G. \end{aligned} \tag{2.13}$$

Thus the scaled TTT mean and the Gini coefficient G are complementary to each other. This representation provides additional insight into G.

In (1.4) and (1.5), I have used the Gini coefficient of the income distribution among the poor and of the income distribution censored at the poverty line, respectively. To define these coefficients properly, we extend the representation in (2.13) to the censored and truncated cases. First, consider the truncated case. Corresponding to α and ω, defined in (1.1), the *truncated distribution function* (at ω) is defined by

$$F_\alpha(x) = \alpha^{-1}F(x), \quad x \in [0, \omega];$$
$$F_\alpha(x) = 1, \quad x > \omega. \tag{2.14}$$

Then let

$$\begin{aligned}\bar{F}_\alpha(x) &= 1 - F_\alpha(x)\\ &= 1 - \alpha^{-1}F(x), \qquad x \in [0, \omega],\\ &= 0, \qquad x > \omega,\end{aligned} \tag{2.15}$$

and let μ_α and G_α be the mean and Gini coefficient for this truncated distribution; that is, $\mu_\alpha = \int_0^\omega y\, dF_\alpha(y)$ and $G_\alpha = 1 - 2\mu_\alpha^{-1}\int_0^\omega y\bar{F}_\alpha(y)\, dF_\alpha(y)$. In the notation of Sen (1976), μ_α represents the mean income of the poor and G_α represents the Gini coefficient of the income distribution among the poor. Then parallel to (2.2) and (2.13), we have

$$\mu_\alpha = \int_0^\infty x\, dF_\alpha(x) = \alpha^{-1}\left\{\int_0^\omega y\, dF(y)\right\} \tag{2.16}$$

and

$$\begin{aligned}\bar{\phi}_\alpha = 1 - G_\alpha &= -\mu_\alpha^{-1}\left\{\int_0^\omega y\, d\bar{F}_\alpha^2(y)\right\}\\ &= 2\mu_\alpha^{-1}\left\{\int_0^\omega y\bar{F}_\alpha(y)\, dF_\alpha(y)\right\}\\ &= \mu_\alpha^{-1}E(Y_{\alpha1} \wedge Y_{\alpha2}),\end{aligned} \tag{2.17}$$

where $Y_{\alpha1}$ and $Y_{\alpha2}$ are two independent random variables, each having the same distribution F_α. Note that by (1.2)

$$\omega^{-1}\mu_\alpha = 1 - \beta. \tag{2.18}$$

In addition, rewrite

$$\mu_\alpha^{-1}E(Y_{\alpha1} \wedge Y_{\alpha2}) = (\omega^{-1}\mu_\alpha)^{-1}E\{(\omega^{-1}Y_{\alpha1}) \wedge (\omega^{-1}Y_{\alpha2})\} \tag{2.19}$$

and note that the $\omega^{-1}Y_{\alpha j}$ are bounded (between 0 and 1) and independent, so $(\omega^{-1}Y_{\alpha1}) \wedge (\omega^{-1}Y_{\alpha2}) \geq (\omega^{-1}Y_{\alpha1})(\omega^{-1}Y_{\alpha2})$; hence by (2.19),

$$\begin{aligned}\mu_\alpha^{-1}E(Y_{\alpha1} \wedge Y_{\alpha2}) &\geq (\omega^{-1}\mu_\alpha)^{-1}E(\omega^{-1}Y_{\alpha1})E(\omega^{-1}Y_{\alpha2})\\ &= \omega^{-1}\mu_\alpha.\end{aligned} \tag{2.20}$$

From (2.17), (2.18), and (2.20), we obtain

$$1 - G_\alpha = \mu_\alpha^{-1}E(Y_{\alpha1} \wedge Y_{\alpha2}) \geq \omega^{-1}\mu_\alpha = 1 - \beta. \tag{2.21}$$

It is easy to verify that in (2.20) the equality sign (in the first step) holds only when the distribution F_α has the entire mass (i.e., unity) at the point 0 or ω. Thus

$G_\alpha \leq \beta$, where the equality sign holds only for F_α being degenerate at the point 0 or ω. (2.22)

This inequality will be of considerable use in the next section for the study of the properties of the poverty indexes.

Consider next the censored case, and define the *censored distribution function* (at ω) by

$$\begin{aligned}F_\omega^c(x) &= F(x), \qquad 0 \leq x < \omega\\ &= 1, \qquad x \geq \omega.\end{aligned} \tag{2.23}$$

Then let $\bar{F}_\omega^c(x) = 1 - F_\omega^c(x)$ be the corresponding survival function, and let μ_ω^c and G_ω^c be the mean and Gini coefficient for this censored distribution. Then

$$\begin{aligned}\mu_\omega^c &= \int_0^\infty y\, dF_\omega^c(y) = \int_0^\omega y\, dF(y) + \omega\bar{F}(\omega)\\ &= \alpha\mu_\alpha + \omega(1 - \alpha)\\ &= \alpha\omega(1 - \beta) + \omega(1 - \alpha)\\ &= \omega(1 - \alpha\beta)\end{aligned} \tag{2.24}$$

and

$$G_\omega^c = 1 - (\mu_\omega^c)^{-1}E(Y_{\omega1}^* \wedge Y_{\omega2}^*), \tag{2.25}$$

where $Y_{\omega1}^*$ and $Y_{\omega2}^*$ are two independent random variables, each having the same distribution F_ω^c. Note that the $Y_{\omega j}^*$ are independent and $\omega^{-1}Y_j^*$ are bounded (between 0 and 1) with probability 1. Hence, repeating the same steps as in (2.19)–(2.20), we obtain

$G_\omega^c \leq \alpha\beta$, where the equality sign holds only when F_ω^c is degenerate at the point ω. (2.26)

(Note that F_ω^c has a mass of $1 - \alpha$ or more at the point ω; hence, for $\alpha < 1$, F_ω^c cannot be degenerate at the point 0.) If F_ω^c is degenerate at ω, then $\mu_\alpha = \mu_\omega^c = \omega$, so $\beta = 0$ and hence $G_\omega^c = 0$.

Next I discuss the relationship between G_α and G^c; in this context, the income gap ratio β plays a vital role. First define

$$\begin{aligned}\gamma = \int_0^\omega y\, dF(y)/\mu_\omega^c &= \{\alpha\mu_\alpha\}/\mu_\omega^c\\ &= \alpha(1 - \beta)/(1 - \alpha\beta).\end{aligned} \tag{2.27}$$

For the censored distribution F_ω^c, in view of the probability mass $1 - \alpha$ attached to ω, for the corresponding Lorenz curve (presented in Fig. 2) we have a segment line on $(\alpha, 1]$. In terms of the shaded areas A, B, and C,

$$G_\omega^c = 2(\mathrm{A} + \mathrm{B} + \mathrm{C}). \tag{2.28}$$

In addition, projecting the rectangle $[0, \alpha] \times [0, \gamma]$ onto the unit square,

$$G_\alpha = (\alpha\gamma)^{-1}2\mathrm{A}. \tag{2.29}$$

Note that

$$\mathrm{B} = (\alpha^2 - \alpha\gamma)/2 = \alpha(\alpha - \gamma)/2 \tag{2.30}$$

and

$$\begin{aligned}\mathrm{C} &= (1 - \alpha)(1 - \gamma)/2 - (1 - \alpha)^2/2\\ &= (1 - \alpha)(\alpha - \gamma)/2.\end{aligned} \tag{2.31}$$

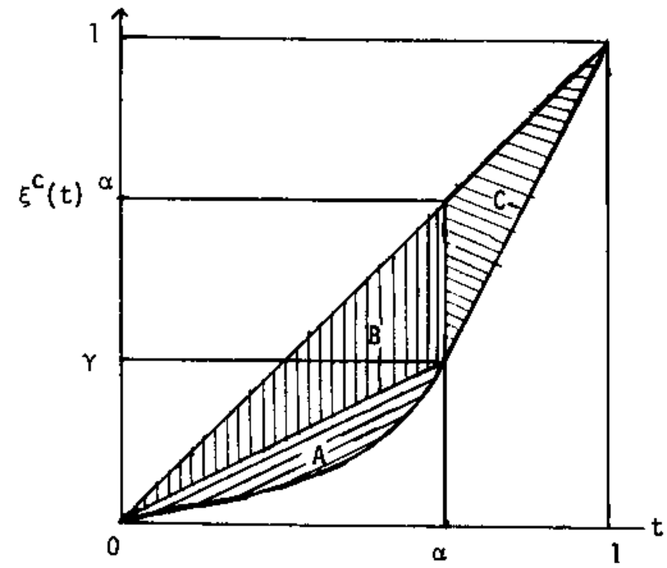

Figure 2. Lorenz Curve for the Censored Income Distribution F_ω^c.

Thus, from (2.28), (2.29), (2.30), and (2.31), we obtain

$$\begin{aligned} G_\omega^c &= \alpha\gamma G_\alpha + \alpha(\alpha - \gamma) + (1 - \alpha)(\alpha - \gamma) \\ &= \alpha\gamma G_\alpha + (\alpha - \gamma) \\ &= \alpha^2(1 - \beta)(1 - \alpha\beta)^{-1}G_\alpha \\ &\quad + \alpha\beta(1 - \alpha)(1 - \alpha\beta)^{-1}, \end{aligned} \tag{2.32}$$

where in the last step I have made use of (2.27). For later use, rewrite (2.32) as

$$G_\omega^c = \alpha G_\alpha + \alpha(1 - \alpha)(1 - \alpha\beta)^{-1}\{\beta - G_\alpha\}. \tag{2.33}$$

Note that by virtue of (2.22), (2.26), and (2.33),

$$\alpha G_\alpha \le G_\omega^c \le \alpha\beta \quad \text{for all censored distributions.} \tag{2.34}$$

The equality signs in (2.34) hold in the degenerate case when α is either 0 or 1, or when $\beta = 0$; in the latter case, the two Gini coefficients are also equal to 0.

3. POVERTY INDEXES: ROBUSTIFICATION

Note that by (1.2), (1.4), and (2.25),

$$\begin{aligned} \pi_S &= \alpha\{\beta + (1 - \beta)G_\alpha\} \\ &= \alpha\{1 - (1 - \beta)(1 - G_\alpha)\} \\ &= \alpha\{1 - \delta_\alpha\}, \text{ say,} \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \delta_\alpha &= (1 - \beta)(1 - G_\alpha) \\ &= \omega^{-1}\mu_\alpha\mu_\alpha^{-1}E(Y_{\alpha1} \wedge Y_{\alpha2}) \\ &= \omega^{-1}E(Y_{\alpha1} \wedge Y_{\alpha2}). \end{aligned} \tag{3.2}$$

This shows that in the Sen poverty index, both $1 - \beta$ and $1 - G_\alpha$ $(=\bar{\phi}_\alpha)$ play an equally important role (though $1 - G_\alpha \ge 1 - \beta$). This feature, however, is not shared by the other two indexes π_A and π_T. Moreover, as β and G_α both lie in the interval [0, 1], (3.1) implies that

$$\pi_S \ge \alpha\beta \quad \text{for all } \alpha\text{: } 0 \le \alpha \le 1 \text{ and } \omega > 0. \tag{3.3}$$

For latter use I set $\pi_T^* = \alpha G_\alpha$ and make use of (2.34) and (3.3). Then

$$0 \le \pi_T^* \le \pi_T \le \pi_A = \alpha\beta \le \pi_S \quad \text{for all } \alpha, \omega. \tag{3.4}$$

Note that although G_α is bounded from above by β, it may be arbitrarily close to 0 (depending on the concentration of the incomes below the poverty lines); hence π_T^* may also be arbitrarily close to 0 (even when β is away from 0). On the other hand, by (2.22) and (3.1),

$$\pi_S \le \alpha\{\beta + (1 - \beta)\beta\} = \alpha\beta(2 - \beta), \tag{3.5}$$

whereas by (2.22) and (3.2),

$$\pi_S \le \alpha\{1 - (1 - \beta)^2\} = \alpha\beta(2 - \beta) \le \alpha \quad \text{for all } 0 \le \alpha, \beta \le 1. \tag{3.6}$$

Thus we have the following ordering of the poverty indexes:

$$0 \le \pi_T^* \le \pi_T \le \pi_A \le \pi_S \le \alpha\beta(2 - \beta) \le \alpha \quad \text{for all } 0 \le \alpha, \beta \le 1. \tag{3.7}$$

With this ordering of the different poverty indexes, next we consider their sensitiveness to various patterns of income inequality below the poverty line. Toward this, note that by (1.3), (1.4), (1.5), (2.22), and (2.33),

$$\pi_S/\pi_A = 1 + (1 - \beta)\beta^{-1}G_\alpha \text{ is contained in } [1, 2 - \beta], \tag{3.8}$$

$$\begin{aligned} \pi_T/\pi_A &= (\alpha\beta)^{-1}G_\omega^c \\ &= \beta^{-1}G_\alpha + (1 - \alpha\beta)^{-1}(1 - \alpha)(1 - \beta^{-1}G_\alpha) \\ &= (1 - \alpha\beta)^{-1}\{(1 - \alpha) + \alpha(1 - \beta)\beta^{-1}G_\alpha\}, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \pi_S/\pi_T = {} &\{(1 - \alpha\beta)[1 + (1 - \beta)\beta^{-1}G_\alpha]\} \\ &\div \{(1 - \alpha) + \alpha(1 - \beta)\beta^{-1}G_\alpha\}. \end{aligned} \tag{3.10}$$

Note that by (3.7), (3.9) can never be greater than 1 and (3.10) is never less than 1. Actually, for $\alpha \le \frac{1}{2}$, a lower bound for (3.10) is $(1 - \alpha\beta)/(1 - \alpha)$ (≥ 1); for $\frac{1}{2} < \alpha \le 1$, a lower bound is $2 - \beta$ for every β: $0 \le \beta \le 1$. In this study of the sensitiveness of these indexes, the key role is, therefore, played by the factor

$$\gamma^* = (1 - \beta)\beta^{-1}G_\alpha. \tag{3.11}$$

First, consider the case of β close to 1, that is, $\omega^{-1}\mu_\alpha$ close to 0. Using (2.22), we conclude that (3.11) converges to 0 as $\beta \uparrow 1$, so in this case all of (3.8), (3.9), and (3.10) converge to the common limit 1. Thus, with a high degree of poverty, as reflected by a large income gap ratio β, all three indexes, π_S, π_T, and π_A, behave quite similarly (and so does π_T^*), although π_S dominates the others. Consequently, when the average income of the poor (i.e., μ_α) is quite small compared to the poverty line ω, all of the indexes yield comparable pictures. This harmony is not true, however, for the more usual case in which β is away from 1. Toward this, next consider the case in which

$$0 < \alpha < 1, \quad 0 < \beta < 1, \quad G_\alpha = 0. \tag{3.12}$$

To emphasize this situation, consider the case in which the income distribution F has a mass α at some point $y_o : 0 < y_o \leq \omega$ and $\Pr(Y > \omega) = 1 - \alpha$; this may arise because of some welfare system providing unemployment benefits but not quite matching with the poverty line ω. In this case, by reference to (2.17), it is easy to show that $\beta = 1 - \omega^{-1}y_o$ (<1); if y_o approaches ω, then both β and G_α also approach 0. From (3.8), (3.9), (3.10), and (3.12), we have, under (3.12),

$$\begin{aligned} \pi_S/\pi_A &= 1 \\ \pi_T/\pi_A &= (1 - \alpha)/(1 - \alpha\beta) \\ \pi_S/\pi_T &= (1 - \alpha\beta)/(1 - \alpha). \end{aligned} \tag{3.13}$$

This shows that if α is close to 1 and β is not, then π_T/π_A may be quite small and π_S/π_T may be quite large. Actually, π_T/π_A is a decreasing function of α and π_S/π_T is increasing in α [for any (fixed) $\beta : 0 \leq \beta < 1$] and $\lim_{\alpha\uparrow 1} \pi_T/\pi_A = 0$ and $\lim_{\alpha\uparrow 1} \pi_S/\pi_T = +\infty$. Keeping in mind the usual interpretation of G_α, we may then conclude that for heavy concentration of incomes below the poverty line and for large proportions of people below the poverty line, π_S and π_T may behave quite differently. Although this differential picture is more predominant for β close to 0, we have a similar picture for intermediate values of β. In addition, for $G_\alpha = 0$, the Sen poverty index is equal to π_A (satisfying Sen's axiom N) but the Takayama index π_T may be arbitrarily smaller than π_A, depending on the income gap ratio β. From this point of view, π_T appears to be less attractive than π_S.

In reality, G_α may not be exactly equal to 0 [so (3.12) may not hold], although under heavy unemployment or larger income gap ratios, one may expect $\beta^{-1}G_\alpha$ to be rather small (compared with 1). In such a case, (3.9) may still be somewhat less than 1 and (3.10) may be considerably greater than 1; (3.8) may exceed 1 only by moderate amounts. This may raise the question of adequacy of the indexes π_A, π_S, and π_T when β is not very close to 0 (or 1) and/or $\beta^{-1}G_\alpha$ is not that close to 1. In practice, mostly, β lies in the middle of the interval (0, 1) and $\beta^{-1}G_\alpha$ is somewhat smaller than 1 [as would be anticipated from (2.17)] so we really need to address this middle-line situation more carefully.

Note that π_T depends solely on G_ω^c; π_A on α, β; and π_S on α, β, and G_α, where G_ω^c is dependent on α, β, and G_α as well. I would like to explore the scope for adaptation on these indexes, and in this respect the Sen index appears to be the most convenient one to be adapted. As such, I will construct a robust variant of π_S. Note that by (3.6), π_S is bounded from above by α and from below by π_A, whereas for $\beta = 1$ these two bounds are the same. For β close to 0, π_S moves more toward π_A; for β closer to 1, the upper bound is closer. We have also noticed in (2.22) that G_α is never greater than β. Moreover, we may rewrite (1.4) as

$$\pi_S = \alpha\{G_\alpha + (1 - G_\alpha)\beta\} = \alpha G_\alpha + \pi_A(1 - G_\alpha). \tag{3.14}$$

Thus π_S is a weighted arithmetic mean of the two bounds α and π_A, where the relative weights are given by the Gini coefficient G_α (for the truncated distribution F_α) and its complement. I do not want to change the role of the Gini coefficient in this context, but I advocate the use of the geometric mean instead of the arithmetic mean (to reduce the inflationary effect) and thereby propose the following index related to π_S:

$$\begin{aligned} \pi^* &= \text{weighted geometric mean of } \alpha \text{ and } \pi_A \\ &= \alpha^{G_\alpha}\pi_A^{1-G_\alpha} = \alpha\{\beta^{1-G_\alpha}\}. \end{aligned} \tag{3.15}$$

Note that by the basic properties of a geometric mean,

$$\alpha \wedge \pi_A = \pi_A \leq \pi^* \leq \alpha = \alpha \vee \pi_A, \tag{3.16}$$

where both of the inequality signs are strict for $0 < \beta < 1$ and both are equality signs for $\beta = 1$. Using the well-known inequality between the (weighted) arithmetic and geometric means of nonnegative numbers, we readily obtain from (3.14) and (3.15) that for all censored distributions (i.e., for $0 < \alpha < 1$)

$$\pi_S \geq \pi^*,$$

$$\text{where the inequality sign holds when } \beta = 1. \tag{3.17}$$

Combining (3.7), (3.16), and (3.17), we conclude that

$$0 \leq \pi_T^* \leq \pi_T \leq \pi_A \leq \pi^* \leq \pi_S \leq \alpha\beta(2 - \beta) \leq \alpha \leq 1 \tag{3.18}$$

for all censored distributions, that is, for every $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. It is the median ordering of π^* that makes it more appealing as an index of poverty.

Having π^* as a plausible index of poverty, we need to study its behavior under (3.12) and other income inequality patterns. Toward this, first note that under (3.12) (i.e., $G_\alpha = 0$), $\pi^* = \alpha\beta = \pi_A$; hence, like the Sen index π_S, π^* is also equal to π_A under full concentration of the poor income (and satisfies axiom N of Sen 1976). Further,

$$\pi^*/\pi_A = \beta^{-G_\alpha}, \quad 0 \leq \beta \leq 1, \quad 0 \leq G_\alpha \leq \beta. \tag{3.19}$$

Hence for every $\beta : 0 \leq \beta \leq 1$,

$$1 \leq \pi^*/\pi_A = (\beta^{-\beta})^{\beta^{-1}G_\alpha} \leq \beta^{-\beta}$$

$$\text{for every } \alpha : 0 \leq \alpha \leq 1. \tag{3.20}$$

Further, the right side of (3.20) converges to 1 as β goes to either 0 or 1, so like π_S and unlike π_T, π^* has good convergence properties for the extreme income gap ratio cases. On the other hand, for the central part (of plausible income gap ratio values), note that by (3.19),

$$\log(\pi^*/\pi_A) = -G_\alpha(\log \beta) = -\beta^{-1}G_\alpha(\beta \log \beta), \tag{3.21}$$

so by (2.22) and the nature of the function $(-\beta \log \beta)$ on the unit interval (0, 1), I conclude that π^* and π_A behave somewhat differently; in fact, π^* lies between π_A and π_S. For β close to 1, all four indexes, π_T, π_A, π^*, and π_S, behave alike; for β close to 0, π^* behaves more like π_A and π_S (but the other two indexes have poor properties). In the central part of the domain of (α, β), π^* is greater than π_A (and hence π_T) but smaller than π_S. To illustrate this point, consider some typical values of α, β, and G_α, and compare these indexes. Since by (2.22), $G_\alpha \le \beta$, in my consideration, we have limited ourselves to values of G_α in the range $(0, \beta)$.

The conservative character of π_T, especially for higher values of α and lower values of $\beta^{-1}G_\alpha$, is quite clear from Table 1. In addition, π^* emerges clearly as a middle runner between π_A and π_S.

So far, I have provided a picture of the performances of the different indexes on an absolute interpretation. In some instances, however, a relative interpretation is of more importance. For example, when measuring poverty, one may be interested in determining which of two income distributions (corresponding to two different societies or periods of time for a common society) exhibits more poverty, and for this situation the ordinal (ranking) properties of a poverty index may serve the purpose well. Thus, when comparing two poverty indexes, it may be of natural interest to inquire whether they yield the same ranking for both of the income distributions under consideration. Thus we need to characterize a class of income distributions under which the relative ranking of the poverty indexes considered earlier remains the same. That is, if F_1 and F_2 are two income distributions and π_1 and π_2 are two poverty indexes, we would like to know under what conditions the F_i, $\pi_1(F_1) > \pi_1(F_2)$, and an ordering as in (3.18) ensure that $\pi_2(F_1) > \pi_2(F_2)$. Fortunately, the results obtained in this and the earlier sections can be used to answer this question to a satisfactory extent.

First, note that by definitions in (1.4)–(1.6) and (2.33), the indexes π_T, π^*, and π_S all depend on the three parameters α, β, and G_α; π_A in (1.3) is a simple function of α and β only. Each of these parameters is a functional of the associated income distribution F. Thus for clarity of understanding I write

$$\alpha = \alpha(F), \qquad \beta = \beta(F), \qquad G_\alpha = G(\alpha, F). \quad (3.22)$$

Given two income distributions F_1 and F_2, we have the corresponding parameters $\alpha(F_j)$, $\beta(F_j)$, $G(\alpha_j, F_j)$ for $j = 1, 2$. The indexes studied earlier are not all similar functionals of these parameters, and the income distributions may not be ordered simultaneously in all three functionals. This makes it clear that an ordinal picture over two or more poverty indexes or two or more income distributions cannot be drawn unless other side conditions are imposed on these income distributions. To illustrate this basic point, consider the following numerical example. Suppose that $\alpha(F_1) = .5$, $\beta(F_1) = .6$, and $G(\alpha_1, F_1) = .2$ and that $\alpha(F_2) = .6$, $\beta(F_2) = .48$, and $G(\alpha_2, F_2) = .4$. Then we have $\alpha(F_2)/\alpha(F_1) = 1.2$, $\beta(F_1)/\beta(F_2) = 1.25$, $\{\beta(F_1)\}^{1-G(\alpha_1,F_1)}/\{\beta(F_2)\}^{1-G(\alpha_2,F_2)} = 1.0322$, and $[\beta(F_1) + (1 - \beta(F_1)G(\alpha_1, F_1))]/[\beta(F_2) + (1 - \beta(F_2))G(\alpha_2, F_2)] = .9883$. These imply that $\pi_A(F_1) > \pi_A(F_2)$, but $\pi^*(F_1) < \pi^*(F_2)$ and $\pi_S(F_1) < \pi_S(F_2)$. Thus, whereas π^* and π_S are concordant with respect to (F_1, F_2), π_A and π^* (and π_A and π_S) are not so. In a second example, let $\alpha(F_1) = .5$, $\beta(F_1) = .34$, $\alpha(F_2) = .6$, $\beta(F_2) = .26$, and $G(\alpha_1, F_1) = G(\alpha_2, F_2) = .20$. Then $\pi^*(F_2) < \pi^*(F_1)$ but $\pi_S(F_2) > \pi_S(F_1)$, so here π_S and π^* are discordant for (F_1, F_2). A similar picture can be drawn for π_T. Thus at best we can look for certain partial ordering of the F_j under which the ordinality of the different poverty indexes is preserved. This partial ordering can most conveniently be described in terms of the three functionals in (3.22).

Table 1. The Values of π_S, π^*, π_A, and π_T for Some Typical $(\alpha, \beta, G_\alpha)$

α	β	G_α	π_T	π_A	π^*	π_S
.20	.10	.05	.0182	.0200	.0224	.0290
		.10	.0200	.0200	.0252	.0380
	.25	.10	.0452	.0500	.0574	.0650
		.20	.0484	.0500	.0660	.0800
	.50	.20	.0938	.1000	.1149	.1200
		.30	.0956	.1000	.1231	.1300
		.40	.0978	.1000	.1320	.1400
	.80	.25	.1548	.1600	.1692	.1700
		.50	.1571	.1600	.1789	.1800
		.60	.1573	.1600	.1829	.1840
.40	.20	.10	.0661	.0800	.0940	.1120
		.15	.0730	.0800	.1018	.1280
	.40	.10	.1257	.1600	.1754	.1840
		.25	.1429	.1600	.2012	.2200
		.30	.1486	.1600	.2106	.2320
	.60	.20	.2063	.2400	.2658	.2720
		.35	.2189	.2400	.2870	.2960
		.50	.2315	.2400	.3098	.3200
	.80	.20	.2918	.3200	.3346	.3360
		.40	.3012	.3200	.3499	.3520
		.60	.3106	.3200	.3658	.3680
.60	.20	.10	.0873	.1200	.1410	.1680
		.15	.1036	.1200	.1528	.1920
	.40	.10	.1547	.2400	.2630	.2760
		.25	.1974	.2400	.3018	.3300
		.30	.2116	.2400	.3159	.3480
	.60	.20	.2700	.3600	.3987	.4080
		.35	.3038	.3600	.4305	.4440
		.50	.3375	.3600	.4648	.4800
	.80	.20	.3969	.4800	.5019	.5040
		.40	.4246	.4800	.5248	.5280
		.60	.4523	.4800	.5488	.5520
.80	.20	.10	.0990	.1600	.1879	.2240
		.15	.1295	.1600	.2425	.2560
	.40	.10	.1506	.3200	.3507	.3680
		.25	.2353	.3200	.4024	.4400
		.30	.2635	.3200	.4212	.4640
	.50	.20	.2400	.4000	.4595	.4800
		.35	.3200	.4000	.5098	.5400
		.50	.0400	.4000	.5657	.6000
	.75	.20	.3800	.6000	.6355	.6400
		.40	.4600	.6000	.6732	.6800
		.60	.5400	.6000	.7130	.7200
		.75	.6000	.6000	.7449	.7500
.90	.10	.10	.0900	.0900	.1133	.1710
	.20	.10	.1010	.1800	.2114	.2520
		.20	.1800	.1800	.2484	.3240
	.35	.15	.1613	.3150	.3687	.4028
		.30	.2759	.3150	.4316	.4905
	.50	.20	.2291	.4500	.5169	.5400
		.40	.3764	.4500	.5938	.6300
	.75	.20	.3323	.6750	.7150	.7200
		.40	.4569	.6750	.7573	.7650
		.60	.5815	.6750	.8021	.8100
		.75	.6750	.6750	.8375	.8438

Definition. *Two income distributions F_1 and F_2 are said to be poverty-index ordered—that is, $F_1 \overset{PI}{>} F_2$—if $(\alpha(F_1), \beta(F_1), G(\alpha_1, F_1)) \geq (\alpha(F_2), \beta(F_2), G(\alpha_2, F_2))$.* (Note that $\mathbf{x} \geq \mathbf{y}$ if the inequality holds coordinatewise.)

Now, looking at (1.3), (1.4), and (1.6), observe that for each of π_A, π_S, and π^*, the partial derivatives with respect to α, β, and G_α are all nonnegative. Moreover, using (2.33),

$$\begin{aligned}\partial G_\omega^c/\partial\alpha &= 2\alpha(1-\beta)(1-\alpha\beta)^{-1}G_\alpha \\ &\quad + \alpha^2\beta(1-\beta)(1-\alpha\beta)^{-2}G_\alpha \\ &\quad + \beta(1-\alpha)(1-\alpha\beta)^{-2} \geq 0 \\ \partial G_\omega^c/\partial\beta &= (1-\alpha\beta)^{-2} \\ &\quad \times \{\alpha(1-\alpha)[1-\alpha G_\alpha]\} \geq 0 \\ \partial G_\omega^c/\partial G_\alpha &= \alpha^2(1-\beta)/(1-\alpha\beta) \geq 0. \qquad (3.23)\end{aligned}$$

Therefore, if we take any two of the indexes π_T, π_A, π^*, and π_S and any two income distributions F_1 and F_2 that are partially ordered in the poverty-index sense, by using the first mean value theorem we obtain (in the trivariate case) that the indexes are concordant; that is, their relative ordering remains the same for both of the income distributions, F_1 and F_2. Thus the poverty-index ordering of income distributions provides a sufficient condition for the invariance of the relative ordering of all of the poverty indexes considered earlier. This poverty-index ordering means that simultaneously with respect to each of the three criteria (proportion of poor, income gap ratio, and the Gini coefficient of the income distribution of the poor), the two distributions F_1 and F_2 are ordered. The partial ordering defined in terms of the Lorenz curves of income distributions (i.e., with respect to Fig. 1, one Lorenz curve lying above the other) is known as the *Lorenz ordering* of distribution functions. In view of the fact that the poverty lines (say, ω_1 and ω_2) for the two income distributions F_1 and F_2 need not be the same (nor are they determined generally by equating the two proportions of poor people or their mean incomes), the Lorenz ordering of F_1, F_2 may not yield the poverty-index ordering. In addition, the two numerical examples considered earlier show that Lorenz ordering may not suffice to preserve the ranking invariance of the different poverty indexes considered here. Recalling that any variation in either of the three parameters in (3.22) may have an influence on the poverty indexes, it seems that a partial ordering preserving some coherence of these variations should be desirable for an ordering of the distributions, and the proposed partial ordering with respect to the vector of parameters in (3.22) seems to be in the right direction.

4. SOME GENERAL REMARKS

To stress the robustness aspects of π^* a bit more, consider a hypothetical situation in which the truncated distribution F_α has two clusters at 0 and ω with respective probability masses $1-p$ and p for some $p: 0 < p < 1$. By (2.17), $G_\alpha = 1-p$ and $\beta = 1-p$, so $\pi_A = \alpha(1-p)$, $\pi_S = \alpha(1-p^2)$, and $\pi^* = \alpha(1-p)^p$. Further, by (2.33), $\pi_T = \alpha\beta = \alpha(1-p) = \pi_A$. Thus in this extreme case π_T agrees with π_A; $\pi_S/\pi_A = 1 + p$ may be considerably larger depending on the value of p. In addition, $\pi^*/\pi_A = \{(1-p)^{1-p}\}^{-1}$ is less affected by the fluctuation of p than π_S/π_A, and as p goes to 1, π^*/π_A converges to 1 (whereas π_S/π_A goes to 2). Consider the other case in which the income distribution among the poor is uniform over $(0, \omega)$; that is, $F_\alpha(x) = x/\omega$ for $0 \leq x \leq \omega$ and $F_\alpha(x) = 1$ for $x \geq \omega$. In this case, $\beta = \frac{1}{2}$ and $G_\alpha = \frac{1}{6}$, so $\beta^{-1}G_\alpha = \frac{1}{3}$. In this case, $\alpha^{-1}G_\omega^c = (6-5\alpha)/6(2-\alpha)$ and is nonincreasing in $\alpha \in (0, 1)$; at $\alpha = 0$ it is equal to $\frac{1}{2}$, and at $\alpha = 1$ it reduces to $\frac{1}{6}$. Thus $\pi_T/\pi_A = (6-5\alpha)/(6-3\alpha)$ is nonincreasing in α; at $\alpha = 0$ it is equal to 1, and at $\alpha = 1$ it reduces to $\frac{1}{3}$. Moreover, here $\pi_S/\pi_A = \frac{7}{6}$ for all $\alpha \in (0, 1)$. Further, $\pi^*/\pi_A = 2^{1/6} = 1.1225 < \frac{7}{6} = 1.1667$. This explains why π^* is less affected than π_S in both of the cases treated before.

In this context, recall that $\pi_A = \alpha\beta$ is dependent on the income gap ratio β and α, where β is not very sensitive to the income inequality (among the poor). Thus π_A by itself may not be an ideal poverty index. The fact that $\pi_T \leq \pi_A$ for all censored distributions suggests that π_T is not only conservative but is subject to the same criticisms. In this respect, both π_S and π^* are more plausible (and both reduce to π_A when $G_\alpha = 0$); in between the two, π^* is less fluctuative than π_S. On this ground, I recommend π^* as a proper poverty index.

Blackorby and Donaldson (1980) considered another normative approach to the formulation of a poverty index. Their proposal relates to

$$\pi_{BD} = f(\alpha, \beta^*), \qquad (4.1)$$

with suitable side conditions on the function $f(\cdot)$ on the unit square $[0, 1]^2$, where β^*, the *representative income gap ratio*, is a distribution-adjusted parameter. In particular, π_A corresponds to the case in which $\beta^* = \beta$ and $f(x, y) = xy$. Similarly, π_S corresponds to the case in which $\beta^* = \beta + (1-\beta)G_\alpha$ and $f(x, y) = xy$; π^* corresponds to the case of $f(x, y) = xy$ with $\beta^* = \beta^{1-G_\alpha}$. This suggests that a poverty index of the form $\alpha\beta^*$, with a suitable β^*, satisfies the normative requirements of Blackorby and Donaldson (1980) and at the same time will be operationally quite convenient. The proposed π^* can also be justified on such a ground.

In Section 2, for the Lorenz curves in the uncensored and censored cases, I treated F as a continuous distribution function. For income distributions in grouped class intervals or for discrete distributions, as is generally the case in practice, the smooth Lorenz curve has to be replaced by segmented lines (see, e.g., Mehran 1975). This may also call for a small correction to the Sen (1976) index π_S, where G_α has to be replaced by $(m+1)^{-1}mG_\alpha$, m being the number of poor people. In the usual case when m is large, however, this hardly makes any real difference in the picture. To be consistent, I also suggest that in the exponent of (3.15) we replace $1 - G_\alpha$ by $1-(m+1)^{-1}mG_\alpha$.

This section concludes with a remark on an alternative measure to the Gini coefficient, suggested by Gastwirth (1975). This coefficient is defined as

$$G^* = E\{|Y_1 - Y_2|/(Y_1 + Y_2)\}, \qquad (4.2)$$

where Y_1 and Y_2 are independent random variables, each having the same income distribution F. The definition immediately extends to the case of truncated or censored distributions. As such, if we define G_α^* for the truncated case, then we may also consider an alternative poverty index wherein in (3.15) we replace G_α by G_α^*. This index will also have similar robustness properties, although the relative picture with these two versions would depend on the underlying F_α.

[*Received October 1985. Revised May 1986.*]

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